EIGHTH REGULAR MEETING OF THE OHIO SECTION.

The eighth regular meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on March 30, 1923, in connection with the meetings of the Ohio College Association and allied societies. Chairman H. L. Coar presided, being relieved by Professor J. H. Weaver for an interval.

Sixty-three persons were registered, the following forty-two being members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, Grace M. Bareis, W. S. Beckwith, R. D. Bohannan, R. L. Borger, J. R. Brandeberry, W. D. Cairns, V. B. Caris, E. H. Clarke, H. L. Coar, O. L. Dustheimer, T. M. Focke, B. C. Glover, H. Hancock, G. P. Harmount, H. W. Kuhn, H. B. Lemon, Anna D. Lewis, L. P. Loomis, E. S. Manson, C. N. Mills, C. N. Moore, C. C. Morris, M. A. Nordgaard, J. R. Overman, A. D. Pitcher, S. E. Rasor, P. L. Rea, Bernice Sanders, Hazel E. Schoonmaker, W. G. Simon, S. A. Singer, C. E. Stout, M. O. Tripp, J. H. Weaver, R. N. Wildermuth, F. B. Wiley, C. O. Williamson, C. H. Yeaton.

At the business session, the secretary reported a membership of eighty-five and ten institutional members as against seventy-eight and nine, respectively, last year. Officers elected for this year are: Chairman, Professor W. E. Anderson, Miami University; Secretary-Treasurer, Professor G. N. Armstrong, Ohio Wesleyan University; Third member of the executive committee, Professor V. B. Caris, Ohio State University. A program committee with continuity of organization was created, Professor Kuhn being chosen for three years service, Professor C. N. Moore, two years, and Professor R. B. Allen, one year. Following an inquiry from the University of West Virginia, it was voted to invite the mathematicians of that state to join in the meetings of the Ohio section. A collection amounting to $19.75 was taken for the uses of the section. The financial situation of the section is satisfactory.

The section dinner, with fifty present, served in the dining room of Campbell Hall, was very successful. The evening session was held in the same room.

The following eleven numbers were presented at the two sessions:

(1) Chairman's address: "Freshman mathematics in the liberal arts college" by Professor H. L. Coar.

Discussion: Professor W. D. Cairns.

(2) "On the tangency of circles" by Professor C. N. Mills.

(3) "Some short methods for the solution of certain differential equations" by Professor R. D. Bohannan.

(4) "Origin and development of our present method of extraction of cube and square roots" by Professor M. A. Nordgaard.

(5) "Industrial mathematics" by Professor J. B. Brandeberry.

(6) "Formulas pertaining to friction on helical gears" by Professor C. O. Williamson.

(7) "Collegiate Euclidean geometry" by Professor M. O. TRIPP.

(8) "The DuBois-Reymond versus the Cantor-Dedekind theory of the irrational number" by Professor S. E. RASOR.

(9) Reports of the Yanney committees on the mathematical situation in Ohio. Professor C. N. MOORE, general chairman.

(10) Discussion: "Why elect mathematics?"

(11) "Number of hours required for a mathematical major in Ohio colleges" by Professor C. N. MILLS.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In his paper on freshman mathematics Professor Coar said: Recent years have seen considerable progress in the teaching of mathematics in college, especially along the line of bringing mathematics into touch with science, business, and life in general. That freshmen in our liberal arts colleges are expected to continue at once the task of acquiring further mathematical technique, without having an opportunity to pause and first secure some mathematical perspective, seems psychologically unsound and is one of the chief reasons why such a small and apparently decreasing per cent. of students elect mathematics. There seems, in this respect, to exist a gap between high school and college mathematics. Lack of some clear knowledge of our elementary number system, too little familiarity with the language of mathematics and its interpretation, a rather hazy conception of how to think analytically, are a few of the elements that constitute this gap. The first semester of the freshman mathematics should be largely devoted to the bridging of this gap. Little, if any, new subject matter should be taught, but the subject matter of the elementary schools will serve as a background for the course. New interest can thus be aroused, and the student will be able to pursue advanced work with greater intelligence and hence with enhanced pleasure and more success.

Differing from Professor Coar in his estimate of the present status, Professor Cairns held that while there is relatively too much purely formal work in high school courses it does not follow that the whole of the present trend and the present experimenting is wrong. This plan would probably clear up the meaning and the bearings of secondary mathematics for the general student, but this would be at the expense of some of the subject matter of the freshman courses and would entirely shut out one of the commendable elements in present-day progress, viz., an introduction to the calculus. Teaching pupils how to think (which, to have any significance, must mean how to think in some particular fields), how to study, and how to express their reasoning clearly is now being done if teachers are at all proficient, and will not be notably improved by the substitution of partially familiar material. One great difficulty is to teach students to avail themselves of the great advantage of any symbolic language and yet to make sure that they think logically and state their arguments clearly. In the face of these criticisms, Professor Cairns expressed his conviction that the proposed plan holds large values potentially and his belief that Professor Coar's

concrete and complete formulation of his plans will offer a valuable contribution to the enlivening of college teaching.

2. Professor Mills presented a method to find the radius of a circle tangent internally to three given circles. Using the cosine theorem to find the cosine of the sum of two angles, the author arrived at a fourth degree equation. The positive root of this equation is the result required.

Note: The author also gave a solution which was being presented on the same day at the Colorado meeting by Mr. J. Q. McNatt.

3. In discussing some short methods in certain types of differential equations, Professor Bohannan showed that the "principle of continuity" as used by D'Alembert in handling linear differential equations with constant coefficients, when the roots are equal, can be extended to all the failing cases when solutions are attempted by the "symbolic method"; as also to the failing cases when the solutions in the general case take the form of series, the failing cases occurring here also when the roots of certain equations are equal or differ by a multiple of the difference of the exponents of consecutive terms of the series.

4. Professor Nordgaard stated that: Our present method of extracting arithmetical roots by the orderly evolving of the digits of the root (evolution) depends upon the inverse of the expression $a^2 + 2ab + b^2$ and $a^3 + 3a^2b + 3ab^2 + b^3$ and upon our decimal place value notation including zero. By illustrations from the source materials from Theon of Alexandria, Aryabhata, Brahmagupta, Māhāvira, Srīdhara, Bhāskara, Leonardo of Pisa, Peurbach, Chuquet, Gemma Frisius, and Cardan, he traced the gradual crystallization of our present method, including the periodization of the radicand, the trial and complete divisor, and the setting-up *schema*.

5. Professor Brandeberry's paper dealt with certain non-credit courses in mathematics given in evening school at Toledo University. These courses are offered in connection with courses given by other departments, such as the departments of drawing, electricity, etc., and are designed especially for men engaged in the industries. Certain experimental courses were described, and it was stated that as a result of the experiences at Toledo University, the best program to follow consisted of an introductory course in shop arithmetic; a course in formula work, which would enable the men to use formulas ordinarily encountered in their handbooks; a course in trigonometry, wherein is taught the solution of right and isosceles triangles, and the solution of triangles by the law of sines and cosines, by naturals; a course in practical mechanics, with emphasis on graphic solutions; a course in strength of materials given in connection with machine and tool designs.

6. The formula of Reuleaux for the work lost by friction when two gears are meshed together,

$$W_f = m\pi f\left(\frac{1}{N} + \frac{1}{N_1}\right)\frac{\epsilon}{t},$$

where W_f = work lost by friction, m = pressure coefficient, f = coefficient of

friction, N and N_1 are the number of teeth on the two wheels, ϵ = angle of contact, t = pitch, is transformed for helical gears ($N = N_1$) into

$$W_f' = K \frac{(N + 2\cos^3\alpha)\cos\left[\gamma + \sin^{-1}\left(\frac{N\cos\gamma}{N + 2\cos^3\alpha}\right)\right]}{\cos\alpha},$$

where $K = mf/\cos^2\gamma$, γ = constant angle of the involute at the pitch circle, α = angle of the helix. A plot of this shows that W_f' decreases as α increases.

7. The object of Professor Tripp was to offer suggestions for emphasizing geometric instruction in college in such a way as to be of direct benefit to the prospective high school teacher. The writer insisted that geometry is of great educational value because of the splendid training it gives in the use of language; spherical trigonometry and descriptive geometry were mentioned as subjects well adapted to the development of space intuition. A course in synthetic geometry which would serve as a sequel to euclidean plane geometry and an introduction to projective geometry was recommended.

8. Professor Rasor's paper introduced the theorem on the necessary and sufficient condition for the convergence of a sequence of numbers. For the DuBois-Reymond theory this theorem was proved and the real number system considered as identical with the totality of all infinite decimal fractions. If the decimal sequence is periodic, it was shown that the number is rational and conversely, and if the sequence is non-periodic the number is irrational. On the contrary, it was pointed out that the Cantor theory starts with the consideration of sequences of rational numbers and then lays down as a definition of a regular sequence what was taken as a theorem in the other theory.

9. At the 1922 meeting, following the suggestion of Professor B. F. Yanney, chairman, six committees to study and report on the mathematical situation in Ohio were formed, with Professor C. N. Moore general chairman. Four of the committees presented reports covering much painstaking work: "College entrance requirements," Professor E. H. Clarke, chairman; "Teacher training in mathematics," Professor C. L. Arnold, chairman; "College courses," Professor W. E. Anderson, chairman; "High school courses," Professor R. B. Allen, chairman. The section expressed its thanks to these committees for their work and requested them to continue their investigations and report further. The general chairman and sub-chairman were requested to edit the reports and summarize the findings for the files of the secretary so that they may be useful to the section and also to the state department of education.

10. In response to a request from the executive committee, many members had sent in reasons why high school freshmen and college freshmen should consider electing mathematics. These were canvassed briefly and a committee consisting of Professors Armstrong, Arnold and Pitcher was appointed to prepare statements addressed to the two classes named and to undertake to place these before the public school pupils concerned before the close of this school year.

11. The basis of this report was a set of questionnaires sent to twenty-six of

the Ohio colleges and universities. The main point of interest was that the number of hours required for a mathematical major ranges from fifteen to thirty-eight hours. So wide a range indicates that the colleges are not certain as to the basic values of many courses. The author urges that more interest should be taken in geometrical subjects, such as modern geometry and descriptive geometry.

G. N. ARMSTRONG, *Secretary-Treasurer.*

ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION.

The seventh annual meeting of the Rocky Mountain section was held at the University of Colorado, Boulder, Colorado, on March 30 and 31.

The attendance was twenty-four, including the following fourteen members of the Association: I. M. DeLong, G. W. Finley, P. Fitch, J. C. Fitterer, G. W. Gorrell, C. A. Hutchinson, Claribel Kendall, O. C. Lester, G. H. Light, F. H. Loud, S. L. Macdonald, J. Q. McNatt, W. J. Risley, H. E. Russell.

The section voted to accept the invitation of the Colorado Fuel and Iron Co. to hold the next meeting at the Steel Plant in Pueblo, Colorado. Mr. J. Q. MCNATT was elected chairman for this meeting.

At the close of the Friday session, the section was favored with a talk by Dr. SAUL EPSTEEN on certain phases of the Einstein Theory.

The following nine papers were read:

(1) "Certain associativity conditions in linear algebras" by Assistant Professor CLARIBEL KENDALL. (Asst. Professor G. W. SMITH, Collaborator.)

(2) "The curve of the price of lumber" by Professor S. L. MACDONALD.

(3) "The intrinsic equation for a family of curves possessing a certain property" by Professor G. H. LIGHT.

(4) "The bearing which the work in the grades has upon college mathematics" by Professor G. W. GORRELL.

(5) "Parabolic grouping of pythagorean triangles" by Professor W. J. HAZARD.

(6) "To calculate the radius of a circle inscribed in a concave triangle" by Mr. J. Q. MCNATT.

(7) "A nomographic perpetual calendar" by Mr. W. K. NELSON.

(8) "Graphical methods for complex roots" by Professor W. J. HAZARD.

(9) "A game of solitaire for an arithmetician" by Dr. F. H. LOUD.

In the absence of the author, a paper by Professor C. H. SISAM was read by title only. All the papers led to considerable discussion. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

(1) In this paper it was shown that the associativity conditions

$$A_{ijkm} \equiv \Sigma(\gamma_{ijk}\gamma_{kim} - \gamma_{ikm}\gamma_{jik}) = 0 \qquad (i, j, k, m = 1 \cdots n)$$

for a linear algebra of order n are highly redundant. For $n = 2$ eight independent

syzygies among A_{ijkm} have been found. Four of these are linear and four quadratic. In the general case there will be at least $n^4 - n^3$ such relations.

(2) Taking the price of all commodities for the year 1913 as a standard, lumber included, the ratio of the price of lumber to this standard was computed for each year, beginning with 1865. Plotting a curve from the results thus obtained gave as a "most probable" equation that of a straight line. It was found that a more accurate equation was that of a parabola. Upon further investigation it was found that a logarithmic curve was a trifle better.

(3) Professor Light derived the general equation of the curves whose polars with respect to a fixed circle cut off a segment on the normal proportional to the radius of curvature. When the circle becomes a point, the intrinsic equations for logarithmic spirals, cardioids, parabolas, circles, lines and hyperbolas are obtained.

(4) It is the belief of a number of college teachers of mathematics, judged by the expressions at association meetings, that the instruction in high schools is of inferior quality. The writer believed this to be largely unfounded. High school teachers are usually well prepared to do their work. The amount of required work in high school is not sufficient to give to the high school teacher the time necessary to make much impression upon his students.

Grade teachers are not specially prepared in mathematics and are unable to give clear explanations in many instances. They are likely to fall into the practice of mere formal work in arithmetic.

A test was given to college freshmen in which problems were taken from a grade text in arithmetic. Several students did not attempt certain problems and very few attempted any analysis, though explanation was requested. The writer was inclined to believe that the student's customary approach to a problem follows his practice in the grades rather than that required in high school.

(5) In his paper Professor Hazard showed that when such triangles are plotted in the first quadrant with one acute vertex at the origin and with the base and altitude of the triangle as the coördinates of the other acute vertex, then the triangles having a common property have their vertices on a parabola. Triangles with a common unit difference between the hypotenuse and one side lie on one parabola; those with a difference of 9 lie on another parabola, etc. All possible integral differences represent, also, integral sums of the hypotenuse and one side. There are four families of parabolas, confocal at the origin, being the loci of the vertices of triangles having a constant integral odd difference, odd sum, even difference, and even sum of the hypotenuse and one side. Possible odd differences and sums are given by the squares of the odd numbers. Even differences and sums are given by twice the squares of all the integers.

(6) A unique method was shown in this paper for computing the radius of a circle inscribed in the area enclosed by three mutually tangent circles, when the radii of these circles were known.

(7) This paper showed how some equations of three or more variables may be represented very simply by the methods of Nomography, while the cartesian

representation is hard to construct and hard to read. A method was given of constructing a chart for an equation which represents the day of the week as a function of the year, month and day of the month, thus producing a graphical perpetual calendar.

(8) Professor Hazard commented on the fact that most text books of algebra give graphical methods for the real roots of quadratic and other equations, but either state or imply that the graphical method fails in the case of complex roots because the plotted curve does not cross the axis of X. When the roots of a quadratic are represented by $(r \pm ni)$ the vertex of the graph is the point (r, n^2), so it is only necessary to find the square root of the ordinate to have the roots determined. This may be done by the geometrical method. Reference was made to the straight line and circle construction for the real roots of a quadratic, used by L. E. Dickson and attributed by him to D'Ocagne and Lill. It was shown how this construction may be extended to find the complex roots of quadratics. Reference was also made to a construction given by Schultze for complex roots.

(9) This paper dealt with problems in the theory of numbers and considered the examination of primes with a view to the relation as modulus sustained by each to smaller numbers, the exponents to which the latter belong, and similar inquiries bearing upon the properties of repetends in both the decimal and binary systems. The author's examination, beginning with the smallest primes, has reached the prime, 2161.

PHILIP FITCH, *Secretary.*

MARCH MEETING OF THE SOUTHEASTERN SECTION.

The second meeting of the Southeastern Section of the Mathematical Association of America was held at Agnes Scott College, Decatur, Georgia, March 10, 1923. The meeting was held in the college chapel with Professor FIELD presiding.

There were eighty-five present, including the following twenty-four members of the Association:

D. F. Barrow, S. M. Barton, J. B. Coleman, T. R. Eagles, Floyd Field, Tomlinson Fort, Miss Leslie Gaylord, J. P. Hill, A. W. Hobbs, J. W. Hinton, Miss Ruby Hightower, J. F. Messick, Miss Fannie S. Mitchell, R. E. Mitchell, A. B. Morton, I. C. Nichols, M. T. Peed, W. W. Rankin, Jr., D. Rumble, David Eugene Smith, D. M. Smith, R. P. Stephens, A. H. Stevens, Miss Rose Wood.

The following officers were elected: R. P. STEPHENS, chairman; TOMLINSON FORT, vice-chairman; W. W. RANKIN, Jr., secretary-treasurer. The executive committee decided to hold the next meeting at the University of Georgia, Athens, Ga.

Agnes Scott College entertained the Southeastern Section at lunch after the regular program had been completed.

Professor DAVID EUGENE SMITH of Columbia University was the principal speaker. Professor Smith was brought to Agnes Scott College by the College Lecture Association. He addressed the faculty and students and many visitors

on Friday night, March 9, on the subject: "Ten Great Epochs in the Human Development of Mathematics." Preceding the address a banquet was held in the Alumnæ House in honor of Professor Smith and Professor H. E. Slaught; the latter, however, was unable to attend the banquet or the meeting of the Association.

The following program was presented:

(1) "Teaching the History of Mathematics in College" by Professor DAVID EUGENE SMITH.

(2) "Correlation of Mathematics" by Professor J. B. COLEMAN.

(3) "Some Reasons Why I Am a Devotee of Mathematics" (by title) by Professor H. E. SLAUGHT.

(4) "Unified Mathematics for Freshmen" by Professor A. W. HOBBS.

(5) "Present Reforms in College Entrance Requirements in Mathematics" by Professor DAVID EUGENE SMITH.

Abstracts of the papers 1, 2 and 4 are given below:

1. The purpose of such courses are (1) cultural, for those who are interested in mathematics and wish to know something of the growth of the subject; (2) educational, for those who propose to teach mathematics, particularly in the secondary schools and colleges. They tend to show mathematics as a steadily advancing subject and they give encouragement to those who look to its continued improvement. At Columbia University the work is open to seniors and graduates and extends over two years. The first half year is devoted to a general survey; the second half year to the history of the leading topics generally taught, and it extends through the history of the calculus. There are this year seventy enrollments in this class. The second year is devoted to individual work, the class being limited to about eight or ten. Each one in this second course works upon his subject of major interest, and the class period is devoted to reports and discussions. The libraries and private collections of material are unusually good in New York, and the speaker showed some of the equipment available for instruction.

2. Professor Coleman presented a general discussion of the vital subject of correlation, or degree of causality, with particular reference to the Pearson "product-moment" coefficient, where straight line relationship is assumed, and correlation ratios, for non-linear relationship. The Pearson coefficient between mathematical average and general average was calculated for students at each of three institutions. With liberal samples at the University of South Carolina, the University of Virginia, and Columbia University, the coefficients were found to be .475, .412 and .697, respectively, indicating a close relationship. Similar coefficients for those with high mathematical averages were found to be .542, .538 and .604, respectively. Any further inference with respect to students of exceptional mathematical ability would hardly be warranted, because of the limited data and lack of uniform variation of the coefficient with the data furnishing these results.

4. There has been a great deal of trouble among college teachers over the

question of good texts in college algebra. The trouble is due not so much to the texts nor to the poor preparation of the students as to the fact that the subject cannot be justified as a required course for college freshmen. The courses in college algebra as given add very little that is essentially new to the student's training. A great deal of it is review of the work done in the High School with the view to perfecting algebraic technique. Algebraic technique cannot be perfected in this way. There is no purpose nor utility evident in these courses and the students are "fed up" on this type of study. The best thing at this stage, according to Professor Hobbs, is a course in general mathematics that will let them see mathematics at work and not always on parade. To this end the function idea should be emphasized and illustrated in problems of finance and mechanics. These problems should not be introduced in the artificial way in which they appear in the traditional texts. Some elementary mechanics should be a part of the course, so that the students will get the physical significance of their work. The derivatives and integrals of some of the functions should be given along with elementary applications to geometry and mechanics. There is no claim here that students taking such courses will be better prepared to go ahead with advanced courses than if they take the traditional courses but there is the claim that if we are to preserve what mathematics has to contribute to a liberal education we must offer the freshmen something that is closer related to life than are theory of equations and Determinants.

W. W. RANKIN, Jr., *Secretary-Treasurer.*

GRAFTING OF THE THEORY OF LIMITS ON THE CALCULUS OF LEIBNIZ.[1]

By FLORIAN CAJORI, University of California.

Leibniz issued his first publication on the calculus in 1684, Newton in 1693. At that time both used infinitely small quantities which were dropped when comparatively small. It is one of the curiosities in the history of mathematics that this rough procedure was adopted, even though before this time theories of limits had been worked out in geometry by Giovanni Benedetti, S. Stevin, L. Valerius, Gregory St. Vincent and Tacquet, and in higher arithmetic by John Wallis. At one time Leibniz, when hard pressed by one of his critics, approached closely to the theory of limits; even to him the theory of limits seemed to be the nearest bomb-proof place of resort. But he never actually founded his calculus upon that theory.

The earliest prominent worker of continental Europe to suggest the founding of the calculus of Leibniz on the limit concept was D'Alembert. The question arises, from what source did D'Alembert derive his inspiration for that project?

[1] Read at a joint session of the Association with the American Mathematical Society and Section A of the American Association for the Advancement of Science at Cambridge, Mass., December, 1922.

He himself had been brought up on the text books of Reyneau and De l'Hospital; he was familiar with the papers of Leibniz and the Bernoullis. All these sources were based on the use of infinitely small quantities. What predecessors influenced him in the study of limits? We find an answer in D'Alembert's great article "Différentiel" in Diderot's *Encyclopédie*, 1754, where he mentions two publications on limits. One of these is Newton's *Quadratura curvarum* of 1704: D'Alembert interprets Newton's "Prime and ultimate ratios" as being really limits. The second publication is De la Chapelle's *Institutions de géométrie*, 1746, a text-book which enjoyed great popularity in France. Through De la Chapelle, D'Alembert connected with the pre-Newtonian theory of limits, as found in Stevin, Gregory St. Vincent and others.

From De la Chapelle, D'Alembert quotes the theorems that "if two magnitudes are the limits of one and the same quantity, the two magnitudes are equal to each other," also the theorem that the limit of a product equals the product of the limits.

Newton's prime and ultimate ratios do not contemplate primarily one constant which one variable approaches; the prime and ultimate ratios are ratios of two quantities just springing into being or else vanishing. Only secondarily does Newton, in applying his theory, consider in the right member of his equations what we would call the limit of a ratio.[1] Accordingly Newton in his *Quadratura curvarum* of 1704 did not quite present what is ordinarily called the method of limits. He considered the ratio of two quantities each of which was approaching the limit zero, rather than the limit of one quantity that was the ratio of the two quantities. In England it required the talent of Benjamin Robins to transform Newton's fluxions of 1704 into a calculus founded on limits; on the continent this transformation exercised the ability of D'Alembert who laid the emphasis upon the limit of a ratio, and gave only secondary consideration to the ratio of limits that were zero. D'Alembert conceives the relation of infinitesimals to limits in this manner: "One feels that the supposition made here of infinitely small quantities is only for abridging and simplifying the argument, but that at bottom the differential calculus does not necessarily postulate the existence of these quantities; that this calculus really consists only of the algebraic determination of the limit of a ratio."

In the article "Limite" D'Alembert states that one calls a magnitude a limit of another magnitude "when the second may approximate to the first closer than by a given magnitude, however small one may suppose it, without however that the approaching magnitude may surpass the magnitude that it approaches, in such a manner that the difference of such a quantity and its limit is absolutely unassignable."

From the explanation that follows it is evident that D'Alembert's variables did not reach their limits. Let us proceed to consider one of the weak points in D'Alembert's exposition of limits. He takes the parabola $y^2 = ax$ and forms the difference-quotient which in modern notation is $\Delta y/\Delta x = a/(2y + \Delta y)$. As Δy

[1] See Newton's *Tractatus de Quadratura curvarum*, Introductio, where he finds the fluxion of x^n.

approaches zero, the limit of the right member $a/(2y + \Delta y)$ is $a/2y$, which must also be the limit of $\Delta y/\Delta x$. Thus dy/dx is the limit of the ratio of $\Delta y/\Delta x$. "But, it will be said," one must make both $\Delta y = 0$ and $\Delta x = 0$, which gives $0/0 = a/2y$. "What does that mean? I answer (1) that there is nothing absurd in this, for the reason that $0/0$ may be equal to anything you wish and therefore equal to $a/2y$; I answer (2) whatever the limit of the ratio of Δy to Δx, when $\Delta y = 0$, $\Delta x = 0$, this limit is not properly the ratio of 0 to 0, because this does not present a clear idea; one cannot form a ratio of two terms of which both are zero." D'Alembert suggests that the ratio of Δy to Δx be considered in the neighborhood of zero, without letting Δy and Δx become actually zero. Thus in the right member of $\Delta y/\Delta x = a/(2y + \Delta y)$, Δy is zero, but on the left side Δy and Δx do not quite reach zero. Here we have the incongruity which troubled conscientious mathematicians during more than a century. It is the same difficulty as that in the theory of prime and ultimate ratios which Newton saw full well, for in 1687 he said, "it is objected that there is no ultimate ratio of evanescent quantities, because the ratio, before the quantities have vanished, is not ultimate; and when they have vanished, is none." [1] How does Newton meet this, his own unanswerable argument? He does so by an appeal to sensuous intuition: "It might as well be maintained that there is no ultimate velocity of a body arriving at a certain place, when its motion is ended. . . . But the answer is easy; for by ultimate velocity is meant that . . . at the very instant when it arrives." [2] If "instant," as used here, is not an infinitesimal constant, the passage is difficult or impossible of interpretation. "That method," said Lagrange, "has the great inconvenience of considering quantities in the state in which they cease, so to speak, to become quantities; for though we can always well conceive the ratios of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become both nothing at the same time."

Recently the hypothesis has been advanced that Newton's definition is "logically sound, for a fluxion is defined by a *Schnitt*." [3] This explanation does not seem valid, for in effecting the Dedekind cut, the commentator uses a range of ratios which is different from the range actually described by Newton.

It seems to the present writer that Newton's ultimate ratios and D'Alembert's $0/0$ left students of the calculus still in the clutches of the infinitely little. But there was no dropping of quantities because they were infinitely small as compared with others. For that reason both Newton and D'Alembert declared that the infinitesimal constant was not necessary in developing the calculus.

During the thirty years between 1754 and 1784 there appeared on the Continent more than 28 publications on the calculus.[4] It is not always easy to

[1] Newton, *Principia*, Book I, Section I, Lemma XI, Scholium.

[2] The Latin passage is as follows: Per velocitatem ultimam intelligi eam, qua corpus movetur; neque antequam attingit locum ultimum et motus cessat, nequa postea, sed tunc cum attingit.

[3] J. M. Child, *Mathematical Gazette*, vol. 11, 1922, p. 27.

[4] We have made extensive use of G. Vivanti's "Infinitesimalrechnung" in M. Cantor's *Vorlesungen über Geschichte der Mathematik*, Bd. 4, 1908, Abschnitt XXVI; also of Vivanti's *Il Concetto d'Infinitesimo*, Naples, 1901.

classify these with respect to the fundamental principles used. Sometimes there is a fusion of methods. Of the 28 texts a rough approximation to the facts assigns 15 to the infinitesimal method of Leibniz and De l'Hospital, 6 to limits not divorced from infinitesimal constants, 2 to Newtonian fluxions [1] expressed in the notation of Leibniz, 4 to the treatment of dx and dy as zeros, 1 (Lagrange), in 1772, to the use of algebra and function theory. The six authors avowedly using limits are Kästner (1761), Karsten (1760), Tempelhoff (1769), Martini (1761), Frisi (1782–5) and Cousin (1777). Of these Kästner and Cousin were the more important. Kästner did not sever all connection with the infinitely small. He used the limit concept for the purpose of justifying the rules relating to the infinitely small of different orders. It will be noticed that these six names are not among the great leaders of mathematical thought of that time. Euler published in 1755 his very influential *Institutiones calculi differentialis* which looked upon dy and dx as absolute zeros. Lagrange in 1761 defended infinitesimals on the theory of the compensation of errors, and in 1772 first based the calculus on algebra and Taylor's theorem. Nor were the six texts using limits popular and widely used; the most popular were those using infinitesimals. During these 30 years the theory of limits did not attract students of the calculus on the Continent as affording a haven of refuge against the assaults of logicians. And why should it? Was not 0/0 as mysterious as anything that Leibniz and De l'Hospital presented? Nor did the theory of limits, as then presented, altogether free the calculus from the infinitesimal constant.

In 1784 Lagrange induced the Berlin academy, of which he was president, to offer a prize for a satisfactory exposition of the calculus. Evidently he was not fully satisfied with his own production of 1772. The academy demanded "a clear and rigorous theory of what in mathematics is called infinity"; "the higher geometry," continued the academicians, "frequently uses infinitely great and infinitely small quantities; but the ancient scientists have carefully evaded the infinite, and a few celebrated analysts of our day admit that the words infinite magnitude are contradictory. Therefore the Academy demands that it be explained, how from an inconsistent assumption so many correct theorems have been deduced, and that a secure and clear fundamental concept be given which can take the place of the infinite without making the calculus too difficult or too long."

The prize was won by S. A. J. Lhuilier who in an earlier book had used indivisibles, but now (1786) proceeded to modify the ancient method of exhaustion so as to make it a sound foundation for the calculus. We pause a moment to consider the Greek method of exhaustion. This method employed all the machinery of the modern theory of limits as ordinarily presented. Antiphon originated the idea of exhausting the circular area by means of inscribed regular polygons with an ever-increasing number of sides. Upon this Eudoxus founded

[1] The only text on the calculus known to me as written and printed on the continent, which uses the notation of Newton exclusively and develops the subject in the manner of the English mathematicians of the eighteenth century, is the *Fluxie-Rekening* of Arnoldus Bastiaan Strabbe, Amsterdam, 1799.

his method of exhaustion, that was employed later by Euclid and Archimedes. Euclid proves (XII, 2) that two circles are to each other as the squares on their diameters, by inscribing a square, then successively doubling the number of sides a finite number of times until the difference between the circle and the last regular polygon constructed is less than a pre-assigned value. This procedure is part of our modern process of showing that a constant is the limit of a variable. But, strange to say, the Greeks do not pass, as we do, from the variable to its limit. They are like a man who undertakes to cross a ditch by a running broad jump, but who abruptly stops short at the brink and finally arrives at his destination by a tedious circuitous route. They had all the modern machinery, but failed to make the most direct use of it. Certain recent writers[1] declare that the ancients did not possess the idea of a variable capable of taking any one of an unlimited number of values, that their process did not exhaust the area, and that the method of exhaustion is erroneously attributed to the Greeks.[2] This view seems to rest upon a misapprehension. With only a finite number of polygons to choose from, Euclid could not have proved his theorems. For in the epsilon-test the pre-assigned value may be any small value and it might be less than the difference between the circle and the largest of the given polygons; in such a case the *reductio ad absurdum* proof of Euclid would become impossible. Only with an unlimited number of available polygons could Euclid complete his argument. But in that case he is exhausting the circle; the method of exhaustion was truly a Greek possession.

Returning to Lhuilier, we observe that his definition of a limit makes the variable always greater or always less than the limit; the variable could not oscillate to values above and below the limit. However, Lhuilier is more liberal than D'Alembert in disregarding the mooted question whether the limit is reached or not. Lhuilier establishes a procession of theorems on limits with a care never before displayed on the Continent, but some of his demonstrations involving infinite series would now be considered insufficient. He avoids the infinitely small. He starts with an algebraic treatment of differentiation and introduces geometric figures later. He considers lim $\Delta y/\Delta x$ as the limit of one variable, not the ratio of the zero-limits of two variables, and thereby takes an important step in advance of D'Alembert. Lhuilier's diffuse monograph did not greatly impress the mathematicians of the time. There is nothing to indicate that it was read as widely as it deserved to be read.

Eleven years passed. The year 1797 brought three events of importance: Lagrange's publication of his *Théorie des fonctions*, Lazare Carnot's *Réflexions sur la métaphysique du calcul infinitésimal*, and S. F. Lacroix's two treatises on the calculus. All three authors were Frenchmen. Lagrange rejects infinity and infinitesimals and, using only the imperfect algebra of his day, endeavors to

[1] C. R. Wallner, *Bibliotheca Mathematica*, Series 3, vol. 4, 1903, p. 250; H. Suter, *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, vol. 63, 1918, pp. 226–228.

[2] Wallner, *loc. cit.*, p. 252, says: "Es ist dabei wohl zu beachten, dass dieser Begriff des Ausschöpfens vor Gregorius nirgends vorkommt, weshalb das Verfahren der Alten ganz mit Unrecht so häufig als Exhaustionsmethode bezeichnet wird."

establish Taylor's theorem and base the calculus upon it. Interesting is Lagrange's characterization of the methods used by his predecessors. He[1] says that "one knows the difficulties offered by the supposition of the infinitely little," that Euler's interpretation of dx and dy as zeros "reduces their ratio to the vague and unintelligible expression zero divided by zero."[2] In one place Lagrange objects to limits as applied to sub-tangents, for the reason that "the sub-tangent is not strictly the limit of the sub-secant, since nothing prevents the sub-secant from continuing to increase after it has become a sub-tangent."[3] Evidently Lagrange had here in mind a more general law which governs the variable than those allowed in the treatment of limits, for he proceeds to state that "veritable limits are quantities which one can not pass, although one can approach them as near as one wishes." He acknowledges that after the manner of Maclaurin, D'Alembert and others, one can rigorously demonstrate the principles of the calculus with the aid of "limits considered in a particular manner," but he adds that "the kind of metaphysics that one is to employ, if not contrary, is at least foreign to the spirit of analysis which does not need any other metaphysics than that which consists of the first principles and the fundamental operations of calculation."[4] As Camille Jordan[5] expressed it, arithmetic and algebra allow four fundamental operations: addition, subtraction, multiplication and division. Passing to the limit, where a variable is replaced by its limit, is a fifth operation. To Lagrange this fifth operation seemed superfluous, if not actually foreign to the spirit of analysis. Hence, his program to eliminate limits from analysis altogether.

Still more unfriendly to the theory of limits was Lazare Carnot. His famous *Réflexions* were written at a period full of political turmoil. That circumstance explains perhaps why he did not familiarize himself with the work of all of his predecessors—that of Berkeley, Robins, Maclaurin and others. The main point of the first edition of his tract is the "compensation of errors" in the system of Leibniz, an explanation which he believed to be new. As a matter of fact, he was anticipated by Berkeley, Maclaurin, and Lagrange. Carnot discusses limits more from the standpoint of D'Alembert than of Lhuilier; he fails to eliminate that troublesome 0/0. He decides in favor of the infinitesimals of Leibniz. Referring to limits he asks, "shall we prefer a thorny path in which it is difficult to avoid being bewildered, to the plain and easy road by which this analysis conducts us to discoveries?" Carnot's book was widely read, but it was not a great contribution to the better understanding of the philosophy of the calculus.

Friendly toward the theory of limits were Lacroix's two treatises, his large *Traité du calcul différentiel et du calcul intégral* (1797–1800) in three ponderous volumes, and the briefer treatise, the *Traité élémentaire de calcul différentiel et*

[1] Lagrange, *Oeuvres*, vol. 7, p. 325.

[2] *Ibid.*, p. 325.

[3] "parce que rien n'empêche la sous-sécante de croître encore lorsqu'elle est devenue sous-tangente." *Ibid.*, p. 325.

[4] *Ibid.*, p. 326.

[5] C. Jordan, *Cours d'analyse*, vol. I, 3. éd., Paris, 1909, p. 15.

de calcul intégral. In the larger treatise the author gives great prominence to Lagrange's point of view as found in his paper of 1772, but in the introduction Lacroix gives also the method of limits. In the briefer treatise he begins with the theory of limits, a theory which, as he says, "best reconciles brevity with rigor in reasoning." [1]

The briefer text enjoyed much greater popularity than the larger text. It was at once translated into German by Grüson; in 1816 it was translated into English. In France the briefer treatise was used during three fourths of a century; the eighth edition appeared in 1874, edited by C. Hermite and J. A. Serret. In explaining the derivative, Lacroix dwells upon the limit of the ratio, but unfortunately also upon the ratio of the limits when the two limits are zero. He discusses 0/0 also in his *Élémens d'Algèbre* and thus unnecessarily stresses this symbol which was casting disrepute upon the logical foundations of the calculus.

At the opening of the nineteenth century the conceptions of limits, as held by the mathematical public, labored under at least seven defects.

The first criticism is that the fixed infinitesimal frequently appeared in the early nineteenth century treatment of limits. Cauchy,[2] who gave a great impetus to the use of limits, defined in 1821 a limit thus: "When the successive values attributed to a variable approach a fixed value indefinitely, so as to end by differing from it as little as wished, this fixed value is called the limit of all others." The words "approach . . . indefinitely" would be objected to by Whitehead [3] on the ground that they apparently suggest infinitely small quantities. But in the actual determination of limits Cauchy uses (p. 29) the epsilon mode of proof. Cauchy was perfectly clear and avoided infinitesimal constants. The same cannot be said of many other writers. The phrase "approaches indefinitely near" ran through many texts of the last century,[4] a dark stream rendering turbid the courses of mathematical reasoning. B. Williamson [5] uses the theory of limits and at the same time defines the differential dx as "less than any assigned quantity, however small." In this interpretation, Williamson follows L. M. H. Navier's *Leçons d'analyse.* A. N. Whitehead [6] and Bertrand Russell [7] both declare that it was K. Weierstrass who banished the infinitely little from the calculus. Although one must acknowledge the wide influence wielded by Weierstrass, this statement is not historically correct; for as early as the eighteenth century Robins, Maclaurin and Lhuilier demonstrated that the calculus can be built up without the use of the fixed infinitesimal. It is true that there were nineteenth-century texts before Weierstrass that used the theory of limits and yet fondled the mystic infinitesimal, just as there were texts after Weierstrass that yielded to the enticing simplicity of these creatures of more than electronic minuteness.

[1] S. F. Lacroix, *Traité du calcul différentiel et du calcul intégral*, Paris, 1810, vol. I, p. xxxv.
[2] Cauchy, *Cours d'Analyse*, Paris, 1821, p. 4.
[3] A. N. Whitehead, *Introduction to Mathematics*, New York, 1911, p. 226.
[4] See, for instance, W. Whewell, *The Doctrine of Limits*, Cambridge, 1838, p. 18.
[5] B. Williamson, *Differential Calculus*, New York, 1884, p. 10.
[6] A. N. Whitehead, *op. cit.*, p. 226.
[7] Bertrand Russell, *Introduction to Mathematical Philosophy*, Cambridge, 1919, p. 107.

A second defect in the early nineteenth-century treatment of limits was the failure to eliminate 0/0 from the explanation of the fundamental concept of a derivative. Bledsoe,[1] a West Point graduate and Professor at the University of Virginia, pronounced this symbol "the most formidable of all the symbols or enigmas in the differential calculus ··· which has kept thousands from adopting that method." The failure to exclude division by zero and the consequent failure to observe the proper limitations of the assumption that the limit of a quotient of two variables is equal to the quotient of the limits prevailed in many books that appeared even later than the third quarter of the nineteenth century. The Leibnizian notation for the first derivative was peculiarly adapted to encourage this mysticism. Bledsoe in his *Philosophy of Mathematics* criticizes Charles Davies for considering two kinds of zeros and gives this matter prolonged consideration without being able himself to throw much light. The treatment of the first derivative in modern books considers simply the limit of the quotient.

A third defect was the absence of a scientific number-system which made itself felt in the course of the arithmetization of mathematics. From this standpoint a variable involved a series of *numbers*, not of *quantities*. A scientific treatment of variables and limits presupposes a satisfactory theory of real numbers, including the irrational. Such number-systems were worked out through the labors of Méray, Weierstrass, Dedekind and others, since the year 1869.

A fourth defect was a failure to consider the question of the existence of a limit. Fortunately, the early studies related to functions of comparatively simple character, "vernünftige Funktionen," so that correct results were usually obtained.

A fifth imperfection of long standing was the adherence by many writers to the famous but faulty doctrine that "what is true *up to* the limit is true *at* the limit." English and American writers usually attribute this doctrine to William Whewell,[2] but De Morgan[3] ascribes it to S. D. Poisson. It was used in the eighteenth century, without being at that time specifically formulated.[4] It was accepted by DeMorgan and in the United States by Davies and Peck.[5] It was discussed by Bledsoe[6] and defended by Simon Newcomb[7] against the attacks upon it made by C. H. Judson,[8] but Newcomb excluded from it the consideration of descriptive relations and limited it, as did Poisson, to quantitative relations.

A sixth defect of the theory of limits, as it existed in the first half of the nineteenth century, was the failure to observe uniform convergence. The

[1] A. T. Bledsoe, *Philosophy of Mathematics*, Philadelphia, 1867, p. 215.

[2] W. Whewell, *Doctrine of Limits*, Cambridge, 1838, p. 21.

[3] A. De Morgan, *Transactions of the Cambridge Philosophical Society*, vol. 8, Part 2, 1844, p. 192.

[4] See, for instance, Guido Grandi, *Quadratura circuli et hyperbolae*, Pisa, 1703; G. W. Leibniz, *Acta eruditorum*, 1713, vol. 5, Suppl. pp. 266, 267, 269. These writers discuss $1 \div (1+x) = 1-x+x^2 - x^3 + \ldots$, when $x = 1$.

[5] Davies and Peck, *Mathematical Dictionary*, Art. "Infinity."

[6] Bledsoe, *op. cit.*, pp. 35, 36.

[7] S. Newcomb, *Analyst*, vol. 9, Des Moines, Iowa, 1882, pp. 114–115.

[8] C. H. Judson, *Analyst*, vol. 8, 1881, p. 109.

removal of this subtle blemish is due to the keen insight of G. G. Stokes [1] in 1847, P. L. Seidel [2] in 1848 and A. L. Cauchy [3] in 1853.

The last imperfection in the treatment of limits that we mention here was the most serious to the rank and file of mathematicians. There were conflicting opinions on the question whether a variable could reach its limit. There is involved in this a conflict between imagination and thought. Can any thing be true which we cannot imagine? Can there be coherent thinking on processes which transcend the imagination? The question is brought to a sharp issue in Zeno's arguments against motion. Upon it depends the race-track controversy as to whether Achilles ever caught the tortoise. If we admit that there are logical processes of the mind which transcend our imagination, then the tortoise is caught after a brief race, then we can explain motion, we have a mind process which is able to interpret what goes on in nature, we can say with Newton, "these generations really take place in the nature of things and are daily seen in the motion of bodies." [4]

As expressed by Couturat, the nerve of the argument that the limit cannot be reached consists in the axiom which has been accepted as evident by many good heads: "The actual infinite cannot be realized." Any movement must contain an actual infinity of parts, which some thinkers have declared impossible.

In the eighteenth century the question whether the limit can be reached was hotly debated by B. Robins and J. Jurin. D'Alembert declared that the limit was not reached and thereby assumed an attitude which has pedagogical advantages over that which transcends the imagination and thereby perplexes the young mind. In the nineteenth century the matter continued to be debated. Cauchy put no restriction upon variables reaching their limits. In 1817 Bolzano,[5] whose writings did not at the time receive the attention they deserved, was concerned with the limits of continuous functions which attain their limits. In Germany, Klügel [6] places no restriction in his definition, but, in the comments which follow, the variable is pictured as not reaching its limit. In 1871 Hermann Hankel [7] starts out in his article "Grenze" by defining limit so that it is not reached; he uses infinitely small quantities. But to remove certain contradictions, he proceeds to develop a new definition which is free from restriction as to the attainment of the limiting value. In France, J. M. C. Duhamel [8] held that the limit is never actually attained. This same view was adopted in England by De Morgan [9] and Todhunter,[10] and in the United States by Bledsoe.

[1] G. G. Stokes, *Transactions Cambridge Philosophical Society*, vol. 8, 1847, p. 533.

[2] P. L. Seidel, *Abhandlungen der Math.—Phys. Classe d. Bayer. Akademie*, München, vol. 5, 1850, p. 381–393.

[3] A. L. Cauchy, *Comptes Rendus*, vol. 36, Paris, 1853, pp. 454–459.

[4] ". . . hae geneses in rerum natura locum vere habent et in motu corporum quotidie cernuntur." From Newton's *Quadratura curvarum*, 1704, Introduction.

[5] P. E. B. Jourdain, "The Development of the Theory of Transfinite Number," *Archiv der Mathematik und Physik*, vol. 14, 1909, p. 297.

[6] G. S. Klügel, Art. "Grenze" in *Mathematisches Wörterbuch*, Leipzig, 1805, vol. 2.

[7] H. Hankel, art. "Grenze" in *Ersch u. Gruber, Encyklopaedie*, Series I, vol. 90, 1871.

[8] Duhamel, *Éléments de calcul infinitésimal*, vol. 1.

[9] A. De Morgan, "Limit," *Penny Cyclopaedia*, 1839.

[10] I. Todhunter, *Differential Calculus*, 7th Ed., London, 1875, p. 6.

On the other hand, William Whewell [1] in England and Charles Davies,[2] another West Point graduate and professor at that institution and at Columbia, held that the variable does reach its limit. Davies says, "if these two quantities are thus to be separated, how can they be brought under the dominion of a common law, and enter together in the same question?"

I illustrate the difficulties encountered by reciting views expressed in this country only forty years ago. If half a debt is paid, then half the remaining debt, and so on, Simon Newcomb argued that "since the debt is halved at every payment, if there was any payment which discharged the whole remaining debt, the half of a thing would equal the whole of it which is impossible." "This is fallacious by proving too much," says De Volson Wood,[3] "Thus, by precisely the same argument, it may be proved that the minute hand of a watch can never overtake the hour hand, that two intersecting right lines can never intersect, that bodies at rest can never move, etc." Newcomb[4] taught that the limit cannot be reached; hence, he said, uniform motion "entirely removes the problem from the class to which the method of limits applies." De Volson Wood declares that regular polygons inscribed in a circle reach their limit under the following conditions: "If the apothem of an inscribed square should grow in length at a uniform rate, and if polygons of double the number of sides be conceived to be instantly described as the growing line becomes the apothem of an octagon, then of a polygon of 16 sides, and so on, the inscribed polygons will reach the circle in a finite time." These are difficult questions on which not all mathematicians agree even at the present time. These are real difficulties that perplex students, and add mystery to the theory of limits.

Modern authors relieve the student's mind of this perplexity by ignoring altogether the question whether the limit is reached or not. It is now seen that the question is one of pure assumption. If we choose to make the limit a term in the range of the variable, the limit is reachable; otherwise it is not reachable. Thus what was a brain-harrowing question for centuries now causes no more mental strain than does the deciding upon an extra lump of sugar for our tea. But the advances in the theory of limits due to Weierstrass and some of his followers have brought changes far more profound. The very idea of a variable has been remodeled. While capable of performing certain acts, there is one feat it can not perform: *the variable cannot vary*. Of a certain number of prescribed values, it takes, not *every one* collectively, but *any one* disjunctively. In general the concept of a variable moving from one value or position to another is abolished. The modern variable has been stripped of every dynamic idea. No longer does it have to pass over an infinite number of steps in a finite time, or else wearisomely keep on stepping forever and ever. This modern theory of limits does not consider directly the question how the minute hand catches up with the hour hand. There is no attempt to reproduce in thought the image of

[1] W. Whewell, *Doctrine of Limits*, Cambridge, 1838, p. xx, 18.
[2] Charles Davies, *Nature and Utility of Mathematics*, New York, 1873, p. 326.
[3] De Volson Wood, *Analyst*, vol. 9, 1882, p. 80.
[4] S. Newcomb, *Analyst*, vol. 9, 1882, p. 115.

what actually transpires in the motion of physical bodies. Some modern pure mathematicians listen with impatience when attempts are made to harness their theory of limits with time and space, with velocity and acceleration and other motions relating to the physical universe. They do not care whether Achilles won the race against the tortoise or not, for this paradox is due to a lack of precision of statement as to what constitutes the range of the variable. Theirs is a beautiful, but very abstract theory, possessing great logical perfection, involving concepts of "class," "range of variable," "neighborhood," "difference," "sequence." We fear that this great gain has been made at considerable sacrifice of objectivity. These modern pure mathematicians cannot say with Newton: "These generations really take place in the nature of things and are daily seen in the motion of bodies." This modern limit-theory cannot be recommended to beginners of the calculus. The physical notion of "velocity" and the "slope of a curve" must be retained as great aids to the young student.

Some defects of the old theory of limits were truly thorns in the flesh. For that reason it is not strange that the infinitely little constants of Leibniz, admitting of great simplicity of manipulation, were actually preferred by many able nineteenth-century mathematicians. S. D. Poisson [1] used infinitely small quantities in his *Traité de mécanique* and he gave what seemed to him valid reasons for the real existence of such quantities. A. A. Cournot in 1841 gave what has been called the clearest philosophy of infinitesimal analysis that has ever been written.[2] Augustus De Morgan in London, and Bartholomew Price at Oxford believed in infinitesimals. Benjamin Peirce, the greatest pure mathematician of America before 1880, was an enthusiastic champion of the fixed infinitesimal. He said "With all the boasted rigor, the ancient Geometry can indeed lead to no result more accurate, none more to be depended upon, than those of the infinitesimal theory: and I doubt if any well constituted mind, well constituted at least for mathematical investigations, ever reposes with more confidence upon the one than the other." However, the consensus of mathematicians now favors the method of limits as more nearly meeting the ideals of logical rigor. While it has saved the calculus from being considered only a crude system of approximation, we can not deny that the treatment involving limits, even in its modern perfected form, lacks directness and simplicity, as compared with the system of infinitesimals. In fact all experience testifies that it would be a pedagogical mistake for us to confine ourselves to the exclusive use of the derivative and to discard the differential. Lhuilier is the only prominent Continental writer who pursued that course. Nineteenth-century writers using the method of limits have habitually employed differentials as well. That is to say, from the derivative $dy/dx = f'(x)$, they passed freely to the differential expression $dy = f'(x)dx$. But great diversity prevailed on the explanation of the differential. How can it be brought into consistent relation with the derivative? The view that dx

[1] S. D. Poisson, *Traité de mécanique*, 2. Éd., vol. 1, Paris, 1833, p. 14.

[2] Max Simon, *Abhandlungen zur Geschichte der Mathem. Wissenschaften*, vol. 8, Leipzig, 1898, p. 128.

and dy were zeros was not satisfactory. With many writers dx and dy were fixed infinitesimals, the very kind of quantities which the method of limits aimed to eliminate. Some looked upon dx as an arbitrary finite increment of x, and dy as the corresponding increment of $f(x)$—a view that was in conflict with the definition of the first derivative. Only in this century have our text books reached substantial agreement by accepting as satisfactory an explanation given about a century ago by Lacroix and Cauchy. In 1797 Lacroix, after defining a derivative as a limit, defined a differential dy as being only a portion of the increment of the function; namely, that portion represented by the product of $f'(x)$ and dx. Cauchy[1] brings out very clearly that the differentials thus defined have a ratio that is equal to the first derivative; that dy is completely determined when one has found the first derivative and agreed upon the value of dx; that dx is purely arbitrary and may be taken to be a finite constant or a variable infinitesimal.

In brief retrospect, we see on the European continent the grafting of the Newtonian and pre-Newtonian concepts of limits upon the calculus of Leibniz; we see during the nineteenth century the gradual elimination of the mystic features and the hampering restrictions that clustered about the limit concept; we see the changes from a naïve treatment to severe arithmetization. Enough of the modern movement is embodied in our elementary text books to relieve the mind of the embarrassment which the older defective theory engendered.

THE FOUR-COLOR PROBLEM.[2]

By H. R. BRAHANA, University of Illinois.

1. Origin. Of the origin of the problem, A. B. Kempe[3] in 1879 says: "It has been stated somewhere by Professor De Morgan that it has long been known to map-makers as a matter of experience—an experience however probably confined to comparatively simple cases—that four colors will suffice in every case. . . . Whether this statement was one merely of belief, or whether Professor De Morgan, or anyone else, ever gave a proof of it, or a way of coloring any given map, is, I believe, unknown; at all events, no answer has been given to the query to that effect put by Professor Cayley to the London Mathematical Society on June 13, 1878." A later reference to an earlier date is made by Guthrie[4] in which he says, "Some thirty years ago . . . my brother," who had been attending Professor De Morgan's class, "showed me the fact that the greatest necessary number of colors . . . is four." Professor De Morgan was pleased with the result and was in the habit of acknowledging to his classes the source of his information. The brother's proof, however, "did not seem altogether satisfactory to himself."

[1] Cauchy, *Leçons sur le Calcul Différentiel*, Paris, 1829; *Oeuvres*, 2. Serie, vol. 4, p. 289.

[2] For additional references, see *Encyklopädie der Math. Wiss.*, III, AB3; also Errera (*cf.* reference § 5).

[3] *American Journal of Mathematics*, 2 (1879), p. 193.

[4] *Proceedings of the Royal Society of Edinburgh*, 10 (1880), p. 727.

The problem is still unsolved. It has afforded many mathematicians experience and very little else.

2. Statement of the problem. The problem may be stated as follows: Given on the surface of a sphere any map, each region of which is such that any two points on its boundary can be joined by an arc made up wholly of points of the boundary, is it possible to color the map with four colors so that no two regions which touch along an arc have the same color?[1]

3. The problem in general. It is obvious that there would be a similar problem for a surface of genus 1, the anchor ring; another for a surface of genus 2, and so on. A natural conclusion to draw off-hand would be that if the problem in the case of the sphere is difficult, the similar problems would become increasingly more difficult as the value of the genus increases. This conclusion is contrary to experience. The problem has been completely solved for each value of the genus p, between 1 and 6, and the way pointed out for $p > 6$. Heawood[2] solved the problem for $p = 1$, and indicated the method of procedure for each other case. His method is given here because the point of difference between the problem on the sphere and that on a surface of higher genus is interesting. First we need only consider maps on which there are three and only three lines at every vertex—such a vertex is said to be of *degree* 3. For, suppose P (Fig. 1) is a point at which there are n ($n > 3$) lines, we may move one end of a along b to the position indicated by the dotted line. This introduces a new vertex of degree 3 and reduces to $n - 1$ the degree of P. It is evident that if the second map M_1 can be colored with k colors, the first one M may be given the same colors, for on returning from M_1 to M we introduce no new contacts between regions. Now if M_1 contains a vertex of degree n greater than 3, we may obtain an M_2 that has one more vertex of degree 3 and has a vertex of lower degree in place of the vertex of degree n in M_1. This may be continued until there are no vertices of degree greater than 3. The system of vertices and lines joining pairs of vertices is called a *linear graph*. If the vertices are all of the same degree the graph is *regular*, and the degree of a regular graph is the same as the degree of one of its vertices. We may note that in a regular graph of degree 3, $2E = 3V$, where E is the number of lines and V is the number of vertices.

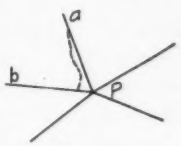


FIG. 1.

Next Heawood obtains a number that he calls the *average number of contacts* for the map. This is $2E/F$ where F is the number of regions in the map. Since each line is on the boundary of two regions, it is evident that if we count the number of sides[3] of each region and add all these numbers, the sum will be $2E$. Now, making use of the generalized Euler formula $V - E + F = 2 - 2p$, where p is the genus of the surface, and of the relation $2E = 3V$, we get an expression

[1] We do not consider the problem of giving the same color to two distinct regions; we consider no countries of more than one region.

[2] *Quarterly Journal of Mathematics*, 24 (1890), p. 332.

[3] By *side* of a region we mean an arc along which it borders a single region.

for the average number of contacts in terms of p and F as follows:

$$\frac{2E}{F} = 6 + 12\frac{(p-1)}{F}.$$

For each value of p it is possible to find a value for F that makes $2E/F$ a maximum. The smallest integer n_p greater than this maximum is evidently a maximum for the number of colors required for any map on a surface of genus p. For, consider the map M of k regions where k is a number such that any map of fewer than k regions can be colored with n_p colors; M must contain at least one region of fewer than n_p sides since n_p is greater than the average. Let a_1^2 be such a region and a_1^1 one of its sides with ends at a_1^0 and a_2^0.[1] Let a_2^1 and a_3^1 be the other two lines with ends at a_1^0, and a_4^1 and a_5^1 the lines with ends at a_2^0. Then by removing the side a_1^1 replacing $a_2^1 a_1^0 a_3^1$ by a line a_6^1, and $a_4^1 a_2^0 a_5^1$ by a line a_7^1 we get a new map whose linear graph is regular and which contains one fewer regions than the original map. This map may therefore be colored with n_p colors; only $n_p - 1$ of them can appear about the region a_1^2, and so we may return the region a_1^2 giving it the n_pth color. Obviously $k > n_p$, so our induction is complete.

The following are values of n_p for small values of p. $n_0 = 6$, $n_1 = 7$, $n_2 = 8$, $n_3 = 9$, $n_4 = 10$, $n_5 = 11$, $n_6 = 12$, $n_7 = 12$, $n_8 = 13$, $n_9 = 13$.

Next Heawood sought a minimum m_p for the number of colors that would be required that one might be assured that any map on a surface of genus p could be colored with m_p colors. His method is to determine a number m_p which is such that the average number of contacts is $m_p - 1$ (in general, a map for which this is true does not have a regular graph). The proof of the validity of m_p for a minimum is complete only when a map of m_p regions each touching each other region on a surface of genus p is given. He gave the map for $p = 1$.[2] The value of m_0 is 4, $m_1 = 7$, $m_2 = 8$, and he showed that this number m_p is the same as n_p for $p > 0$. The sphere is the only surface for which the two do not agree.

4. Kempe's solution. Not only did Heawood treat rather successfully the problem for surfaces of higher genus, but he pointed out an error in the solution of the problem for the sphere that had been given by A. B. Kempe (*cf.* § 1) and that had been generally accepted as correct. Kempe showed that we need consider only maps whose graphs are regular and of degree 3. He introduced the notion of *chains* which has probably come in for consideration by everybody who has worked on the problem since.

The work which follows uses the same considerations as Kempe used but is in a different form. We will use a method of induction. Four colors suffice for any map of four regions. Then either four colors suffice for any map, or there exists a map of k regions that cannot be colored with four colors, where k is such

[1] By this notation we designate with a superscript the dimensionality of the element,—thus, a_i^0 is a point, a_i^1 is an arc, a_i^2 is a region.

[2] Heffter gave the maps for $p = 1, 2, \cdots 6$. *Mathematische Annalen*, 38 (1891), p. 477. Another set of "maps of verification" is given for $p = 1, 2, 3$, by the author in a paper soon to appear.

a number that any map of fewer than k regions can be colored. We investigate this hypothetical map M of k regions.

Now M contains no two-sided region, also no triangular region, for if it does contain one of either kind, we remove one side and combine it with one of the surrounding regions; the resulting map can be colored. No more than 3 colors are used around the region. We introduce it again giving it the fourth color and so get M colored in four colors.

Likewise, it can have no four-sided region. Here we make use of the notion of *chains*. Suppose M has a four-sided region. We remove one side of it combining it with one of the surrounding regions. The resulting map can be colored. The regions surrounding the four-sided one may be colored in order A, B, A, and C, in which case we give the fourth color D to the four-sided region. Or they may be colored A, B, C, and D (Fig. 2). In this case let us consider the group of regions, each region having the color A or the color C, that can be reached starting with region a_1^2 of the figure and adding to it every C region which touches it, then adding each A region that touches one of these C regions, and so on. This group of regions Kempe called an *AC chain*. Two AC chains can have no vertices in common, and no regions in common. No BD chain can cross an AC chain.

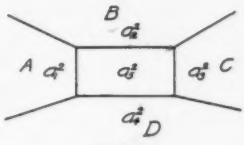


FIG. 2.

Now either the AC chain beginning at a_1^2 contains region a_3^2 or it does not. If it does, then the BD chain beginning at a_2^2 does not contain a_4^2, and if we interchange B and D on that chain we have but three colors about a_5^2 and may give it the color B. In the opposite case we interchange colors A and C on the chain starting at a_1^2 and give a_5^2 the color A.

Next, as Kempe showed, there must be five-sided regions in M. Let a_i be the number of i-sided regions in the map, and suppose there is no region with more than l sides. Then $\sum_{i=2}^{l} a_i$ is the number F. Also, $2E = \sum_{i=2}^{l} i a_i$, for the latter is the sum of the sides of countries, which gives every line twice. Also $2E = 3V$. Using the Euler formula, we get

$$4a_2 + 3a_3 + 2a_4 + a_5 = 12 + \sum_{i=6}^{l} (i - 6)a_i.$$

We have already shown that $a_2 = a_3 = a_4 = 0$, so a_5 is at least 12.[1]

Let us make a short digression to observe that five colors are sufficient for any map on a sphere. It is evident that the minimum map which cannot be colored in five colors must contain a five-sided region. Let one such region a_1^2 be removed as above and let the map be colored. The only possibility that need engage our attention is the case where the regions $a_2^2 \cdots a_6^2$ about a_1^2 have the colors A, B, C, D, and E respectively. The AC chain beginning at a_2^2 contains a_4^2 or does not. In the first case the BD chain beginning at a_3^2 does not

[1] By considering the number of lines, Wernicke (*Mathematische Annalen*, 58 (1901), p. 413) shows that there must be a pentagon adjacent to another pentagon, or else to a hexagon.

contain a_5^2, and after permuting B and D on this chain we may give the color B to a_1^2; in the opposite case we interchange A and C on the AC chain beginning at a_2^2 and give a_1^2 the color A.

Returning to our problem, let a_1^2 be the five-sided region (Fig. 3) surrounded by $a_2^2 \cdots a_6^2$. The only case to consider is the one where $a_2^2 \cdots a_6^2$ take the colors A, B, A, C, and D respectively.

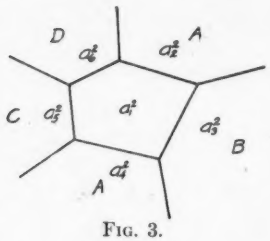


FIG. 3.

First let us notice that the BC chain beginning at a_3^2 must include a_5^2, otherwise we may interchange colors B and C on this chain and then give the color B to a_1^2. Also, the BD chain beginning at a_3^2 must include a_6^2, otherwise we may interchange B and D on it and give a_1^2 the color B. As a result of the first chain it follows that the AD chain which contains a_4^2 does not contain a_6^2 and that by interchanging A and D on it we have two D countries and only one A country touching a_1^2. Or, as a result of the second chain we may interchange A and C on the chain beginning at a_2^2, without changing the color of a_5^2. Kempe interchanged the color on the AD chain, giving a_4^2 the color D, and leaving the others unchanged, *and then* he interchanged the colors on the AC chain, giving a_2^2 the color C and leaving the others unchanged. Thus he was able to give a_1^2 the color A. As Heawood pointed out, he failed to notice that after the first change it is no longer necessary that the BD chain beginning at a_3^1 shall contain a_6^2. If this chain does not contain a_6^2, then an interchange of A and C on the chain beginning at a_2^2 would put C at a_2^2 and A at a_5^2 and there would still be four colors about a_1^2. Heawood not only pointed out this fault in the argument but he presented a map where the difficulty suggested above was actually realized. This map did not, however, disprove the theorem that four colors suffice, for he gave a way of coloring it.

5. The results obtained by use of chains. In a paper entitled *du Coloriage des Cartes et de quelques Questions d'Analysis Situs* [1] Errera carries to its conclusion another method of attack. The conclusion is, as often happens with this problem, an example which shows that the method will not solve the problem. His method is as follows: Make the first interchange of colors as in the paragraph above. This gives the color D to a_4^2. Then we may suppose that there is an AC chain containing both a_2^2 and a_5^2. Interchange B and C on the chain beginning at a_2^2. He continues in this manner; each step, once the first change has been made, is determined. If at any time any of these chains joining certain pairs of regions are missing, the map can be colored. Either that must happen or else the process continues indefinitely. Errera hoped to prove that the latter was impossible.

There are but a finite number of ways of putting four colors on a finite number of regions, so the operation he describes must lead at some stage to the original

[1] Paris, Gauthier-Villars et Cie. (1921).

map. He notes that only after the twentieth operation do the same colors occupy the same positions around a_1^2 as at the beginning. This does not mean that the whole map has taken its original coloring, but rather that the number of operations required to return the map to its original state is a multiple of 20. He hoped to show that no map could have twenty substitutions as the whole or part of its period, and thus that the process would always lead to a map that could be colored. He ended by constructing a map of 17 regions which took on its original coloring after 20 operations. Also he gave a way of coloring the map, thus leaving the problem as much a problem as ever.

In a paper "On the Reducibility of Maps" [1] Birkhoff uses the notion of chains. He defines a *ring* as a set of k regions $a_1a_2 \cdots a_k$ "such that each of these regions has a boundary line in common with the one preceding and following it in cyclical order, but with no other region of the set. A ring R of regions of this kind divides the map into two sets of regions M_1 and M_2 which together with R make up all the regions of the map." He then attempts to color the partial maps $M_1 + R$ and $M_2 + R$ in such a way that the regions of R will have the same colors in both cases or in such a way that by the use of the Kempe chains the regions of R can be given the same colors in both cases. Then putting the partial maps together, one could get the original map colored. He is able to show that the minimum map M (§ 4) contains no rings of four regions; no ring of five regions except about a single region; no ring of six regions surrounding only five-sided regions where the number of such regions is greater than 3; and no ring of n regions within which are n 5-sided regions surrounding a n-sided region; and no ring of $4n$ regions within which are $2n$ 6-sided regions about a single region.

6. Other forms of the problem. There are other interesting forms of the problem. We are considering only the problem of regular maps—maps whose linear graphs are regular and of the third degree, and which contain no 1-, 2-, 3-, or 4-sided regions.

One way to treat the problem is to consider only the linear graph of the map. Tait [2] was the first to notice that the problem was identical with the problem of assigning numbers 1, 2, and 3 to the lines of the graph in such a way that a line bearing each number would be at each vertex.

To see that the two problems are equivalent suppose first that a given map has been colored in four colors and consider the countries at any vertex. If we number the lines according to the following scheme,

$$1 = \frac{A}{B} = \frac{C}{D}$$

$$2 = \frac{A}{C} = \frac{B}{D}$$

$$3 = \frac{A}{D} = \frac{B}{C},$$

[1] *American Journal of Mathematics*, 35 (1913), p. 115.
[2] *Proceedings of the Royal Society of Edinburgh*, 10 (1880), p. 501.

by which we mean that a 1-line is between an A country and a B country or between a C country and a D country, and so on, we see that every vertex will have a 1-line, a 2-line and a 3-line.

Conversely, suppose we have the numbers 1, 2, and 3 attached to the lines in the above-described manner. We may assign the color A to any region a_i^2 and the color of every other region is determined.

To see that no two adjacent regions are given the same color, consider two consecutive sides of the region a_i^2 which was given the color A. One of those sides must be either a 1-side or a 2-side. Such a 1-line or a 2-line will be included in some $(1-2)$ circuit—*i.e.*, a set of lines $l_1 l_2 \cdots l_n$ where l_1 and l_2 have a vertex in common, l_2 and l_3, and so on to l_n and l_1 and where l_1 is a 1-line, l_2 is a 2-line, l_3 is a 1-line, and so on. Such a circuit must contain an even number of lines and an even number of vertices; there must be a 3-line at every vertex of the circuit. All of the 1-lines and 2-lines constitute a set of circuits of the above type. A $(1-2)$ circuit divides the sphere into two regions, one of which may be called the inside of the circuit and the other the outside. There may or may not be $(1-2)$ circuits inside of a given $(1-2)$ circuit C; if there is a single $(1-2)$ circuit C_1 inside of C, then C_1 is connected to C by an even number of 3-lines. For C_1 has an even number of vertices and an even number (which may be zero) of those are joined in pairs by 3-lines. The remaining vertices, of which there are an even number, must be joined to C by 3-lines. If C is a circuit with n circuits $C_1, C_2, \cdots, C_n$ inside, no one of which contains more than one of the others inside it, then C is joined to the system by an even number of 3-lines; for each C_i $(1 \leq i \leq n)$ has an even number of vertices joined by 3-lines to some circuit outside it, and so there will be an even number for the whole system, and each 3-line joining two of them reduces by 2 the number of 3-lines joining the system to C. Thus we see that there are an even number of 3-lines going outward from C.

Now let the circuit which touches the region a_i^2 be C. The region a_i^2 is given the color A. Then color every region that touches the circuit C and that can be reached from a_i^2 by crossing 3-lines. If this method ever leads us back to a_i^2 we must have crossed an even number of 3-lines and so a_i^2 would not be given a color different from A. If this set of regions which has just been colored touches any other $(1-2)$ circuit, we may continue crossing 3-lines and coloring regions A and D without ever coming to a contradiction. When this has been carried as far as possible we will have obtained a Kempe chain. Now cross a 1-line or a 2-line from any region of this chain. According to the table above, this leads to a B region or a C region. From this continue crossing 3-lines as before and assigning colors according to the table. This gives a BC chain. Continuing in this way as long as there are any uncolored regions, we finally arrive at a coloring of the map. Thus we have shown that the two problems are equivalent.

We will now introduce the following definitions:

The number of vertices of a linear graph is called the *order* of the graph.

A regular graph is said to be *factorable* if it is possible to select a set of lines of the graph that constitutes a regular graph of the same order but of lower degree; the set of lines selected is one *factor*, and the remaining set of lines is the other.

A *leaf* is a part of the graph that is connected with the remainder by a single line provided it contains no such part itself.

An affirmative to the map problem is equivalent to the following theorem:

Any regular graph of degree 3 *which contains no leaves and which can be put on a sphere can be factored into three first degree factors.*

Tait, accepting Kempe's solution and believing that he had solved the problem in several ways himself, stated this theorem as a corollary. (*Cf.* reference above.)

The most that has been accomplished in this direction is the statement and proof by Petersen [1] of the following theorem:

Any linear graph containing less than three leaves can be factored into a first degree factor and a second degree factor.

It follows *a fortiori* that the linear graph of a regular map on a sphere can be factored into a linear and a quadratic factor. This second degree factor is made up of a number of circuits, which may, however, contain an odd number of vertices and lines.

We may go a step further in refinement. The problem of attaching numbers 1, 2, and 3 to the lines of the linear graph so that one of each kind is at every vertex is equivalent to the problem of attaching $+1$ or -1 to each vertex so that the sum of the numbers on the vertices of any circuit is equal to zero (mod 3). Heawood [2] was the first to put the problem in this form. Veblen [3] applied modular equations to the problem and so translated it into a problem in finite geometry.

Let the number of vertices be α_0, the number of lines be α_1, and the number of regions be α_2. Suppose we were to attach a number to each region, each number to represent a single color and no two distinct numbers to represent the same color. The condition that two adjacent regions have the same color could be expressed by the modular equation

$$y_a + y_b = 0 \pmod 2, \tag{1}$$

where y_a and y_b are the numbers attached to the regions a and b which touch along a particular line. For every line we may write an equation of the above type. The problem may then be considered as the problem of finding a set of α_2 values $(y_1, y_2, \cdots y_{\alpha_2})$, where y_i may have any one of four values, which does not satisfy any of the α_1 equations (1).

He then takes the field $GF(2)$ consisting of 0 and 1 combined modulo 2 and extends it by the Galois imaginaries [4] satisfying the relation $i^2 + i + 1 = 0$.[5]

[1] *Acta Mathematica*, 15 (1891), p. 193. A second proof was given by the author, *Annals of Mathematics*, (2) 19 (1917), p. 59; and later a proof was given by Errera (*loc. cit.*).

[2] *Quarterly Journal of Mathematics*, 29 (1898), p. 270.

[3] *Annals of Mathematics*, 25, 14 (1912), p. 86.

[4] For a discussion of Galois imaginaries see Dickson: *Introduction to the Theory of Algebraic Equations*, p. 42 ff.

[5] i is no longer used as an index.

The extended field $GF(2^2)$ has four elements 0, 1, i and $i+1$. A set of values $(y_1, y_2, \cdots y_{\alpha_2})$ may be considered as a point on a finite projective space of $\alpha_2 - 1$ dimensions provided we exclude the set $(0, 0, \cdots 0)$ and consider $(ky_1, ky_2, \cdots ky_{\alpha_2})$ to be the same point as $(y_1, y_2, \cdots y_{\alpha_2})$. If the variables y_i range over the $GF(2)$ there will be $2^{\alpha_2} - 1$ points in the space, if they range over the $GF(2^2)$ there will be $(4^{\alpha_2} - 1)/3$ points. The first space is included in the second and the points of the second not included in the first may be regarded as *imaginary* with respect to the first space.

Each of equations (1) represents an $(\alpha_2 - 2)$ space. *Any point which does not lie on any one of these* $(\alpha_2 - 2)$*-spaces represents a solution of the problem.* In general, no real point can represent a solution because then the map would require but two colors which is impossible when the map contains a vertex at which there are an odd number of regions. Every imaginary point is on one and only one real line. But if $(y_1 + iy_1', y_2 + iy_2', \cdots y_{\alpha_2} + iy_{\alpha_2}')$ satisfies one of the above equations, so must $(y_1, y_2, \cdots y_{\alpha_2})$ and $(y_1', y_2', \cdots y_{\alpha_2}')$, and conversely. Hence a solution is given by *each real line which does not lie on any of the* $(\alpha_2 - 2)$*-spaces which are represented by equations* (1).

He also states the problem in terms of modular equations in a form equivalent to the theorem on linear graphs, and still again in a form equivalent to the problem of attaching numbers $+1$ and -1 to the vertices. The only question remaining is whether or not the systems of equations have solutions. Similarly, Birkhoff [1] gives a formula for $P_n(\lambda)$, the number of ways in which a given map of n regions may be colored in λ colors. This, however, does not settle the question, for we have yet to find out whether or not $P_n(4)$ is positive for all values of n.

Finally, Franklin [2] arrives at the rather disheartening conclusion that the most we can say about the minimum uncolorable regular map, taking account of all the reductions that have been made and many more which he makes himself, is that it contains at least 26 regions. He gives a map of 42 regions to which none of the reductions apply. He establishes by a method similar to that used by Birkhoff (see § 5) that the minimum uncolorable regular map does not contain:

(1) A side of a hexagon surrounded by the hexagon and three pentagons;
(2) A pentagon in contact with three pentagons and a hexagon;
(3) A pentagon surrounded by two pentagons and three hexagons;
(4) An even sided region completely surrounded by hexagons and (some or no) pairs of pentagons, the two of each pair being adjacent;
(5) A pentagon surrounded by four pentagons and two hexagons;
(6) A $2n$-sided region surrounded by $2n - 2$ pentagons and two other adjacent regions;
(7) A $(2n - 1)$-sided region surrounded by $2n - 2$ pentagons and one other region.

[1] *Annals of Mathematics*, 25, 14 (1912), p. 42.
[2] *American Journal of Mathematics*, 24 (1922), p. 225.

By a method similar to that used by Wernicke (see footnote, § 4), he shows that, as a result of the above and other like restrictions on the minimum uncolorable map, it contains one of the three arrangements which follow:
(1) A pentagon adjacent to two other pentagons;
(2) A pentagon adjacent to a pentagon and a hexagon;
or
(3) A pentagon adjacent to two hexagons.

A GRAPHICAL TREATMENT OF MIXTURES.

By F. E. WOOD, Northwestern University.

1. Introduction. Mixtures of Two Substances. Let the points of a line segment P_1P_2 of unit length represent mixtures of two materials P_1 and P_2, the mixture of a_1 parts of P_1 and a_2 parts of P_2 being represented by the point whose distance from P_1 toward P_2 is $a_2/(a_1 + a_2)$. Evidently its distance from P_2 will be $a_1/(a_1 + a_2)$, and the points P_1, P_2 will represent 100 per cent. of P_1, P_2 respectively. We take as the *coördinates* of the general point P the values $a_1/(a_1 + a_2)$, $a_2/(a_1 + a_2)$. In order to find the point corresponding to a mixture of a_1 parts of P_1 and a_2 parts of P_2 we divide the segment P_1P_2 by a point M such that $P_1M/MP_2 = a_2/a_1$, the point M being the desired point.

Corresponding to each point between P_1 and P_2 there is a possible mixture of the two materials P_1 and P_2, and conversely for each possible mixture there will correspond a point between P_1 and P_2. To find the mixture corresponding to a point M between P_1 and P_2 we determine the ratio $P_1M/MP_2 = r$; then the mixture is composed of $\rho/(1 + r)$ parts of P_1 and $\rho r/(1 + r)$ parts of P_2 (ρ being a proportionality factor). Two mixtures will have the same corresponding point if and only if they have the same value of a_2/a_1, so that the point corresponding to a mixture depends upon the composition only, and is not affected by the amount of the mixture. Due to this correspondence between mixtures and points on a segment, we shall in the sequel speak of the point $M(a_1, a_2)$ where the coördinates of M are a_1 and a_2 respectively $(a_1 + a_2 = 1)$, and the mixture $M(a_1, a_2)$ where the mixture M is made up of a_1 parts of P_1 and a_2 parts of P_2 and $a_1 + a_2 = 1$; between the point M and the mixture M there is a unique correspondence. To a point upon the line P_1P_2, but not between P_1 and P_2, there would correspond an impossible mixture, since a negative amount of one material would have to be taken.

Consider two mixtures of P_1 and P_2, $M_1(a_1, a_2)$ and $M_2(b_1, b_2)$. Let M denote a mixture of k_1 parts of A and k_2 parts of B, where $k_1 + k_2 = 1$; the mixture $M(m_1, m_2)$ contains $k_1a_1 + k_2b_1$ parts of P_1 and $k_1a_2 + k_2b_2$ parts of P_2. Since $P_1A = a_2$, $P_1M = m_2$, $P_1B = b_2$ we find that $M_1M/MM_2 = k_2/k_1$; therefore *the point corresponding to a mixture of any two mixtures of P_1 and P_2 may be found in the same way as the point corresponding to a mixture of P_1 and P_2, the required*

point being the one which divides the segment corresponding to the two mixtures in the inverse ratio to the parts of each mixture taken.

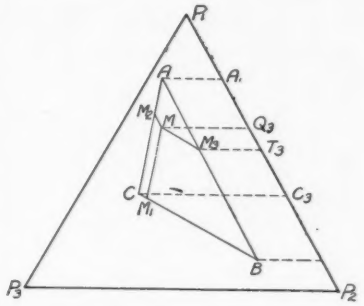
To each point upon the segment M_1M_2 there corresponds a possible mixture M_1 and M_2, and for each point on the line M_1M_2 but not on the segment M_1M_2 there corresponds no possible mixture.

2. Mixtures of Two Diluted Solutions. Some mixtures contain two valuable or important substances and a third unimportant or valueless one (as an ore containing lead, silver and refuse, or a stock food containing protein, carbohydrates and "roughage") where the percentage of the amount of the valuable constituents to the amount of the whole mixture is relatively unimportant. We will denote the substances regarded as valuable "active" and the one which dilutes them "inactive"; in such mixtures only the ratio of the one active substance to the other active substance is important, so that *ratio* mixtures only are to be considered. Two ratio mixtures will be said to be *equivalent* when the percentages of the active substances which enter are in the same ratio for each mixture. The *concentration* of a ratio mixture is defined as the ratio of the amount of active substances to the total amount of the mixture. It is the sum of the percentages of the active substances. The concentration of a mixture containing two undiluted substances is unity. It is evident that an equivalent mixture to any ratio mixture which can be obtained from two substances of concentration unity can also be obtained when the concentrations of the active substances have been altered by the addition of an inactive substance, and conversely. The only changes which arise will be in the formulas.

Let the concentrations of P_1 and P_2 be denoted by α_1 and α_2 respectively; then to a mixture of a_1 parts of P_1 and a_2 parts of P_2 we make to correspond a point M on the unit segment P_1P_2 such that $P_1M = a_2\alpha_2/(a_1\alpha_1 + a_2\alpha_2)$, whence $MP_2 = a_1\alpha_1/(a_1\alpha_1 + a_2\alpha_2)$. For, suppose P_1' and P_2' to be ratio mixtures made from P_1 and P_2 by the addition (or extraction) of the inactive substance to such an extent that the concentrations of P_1' and P_2' are each .01, then one part of P_1 will have as much of the active substances as $100\alpha_1$ parts of P_1', and one part of P_2 as much as $100\alpha_2$ parts of P_2'; so a ratio mixture M' equivalent to M can be made $100a_1\alpha_1$ parts of P_1' and $100a_2\alpha_2$ parts of P_2' whence the coördinates of M' would be $a_1\alpha_1/(a_1\alpha_1 + a_2\alpha_2)$, $a_2\alpha_2/(a_1\alpha_1 + a_2\alpha_2)$. Since the point M' corresponds to all mixtures equivalent to the mixture M', it will coincide with M, whence $P_1M = a_2\alpha_2/(a_1\alpha_1 + a_2\alpha_2)$ as stated.

To determine the point M corresponding to a ratio mixture M of a_1 parts of P_1 of concentration α_1 and a_2 parts of P_2 of concentration α_2 we divide the segment P_1P_2 by a point M such that $P_1M/MP_2 = a_2\alpha_2/a_1\alpha_1$. To determine the ratio mixture which corresponds to a point M upon the segment P_1P_2 we determine the ratio $P_1M/MP_2 = r$, then $a_2\alpha_2/a_1\alpha_1 = r$, $a_1 + a_2 = 1$; so we find $a_1 = \alpha_2/(\alpha_2 + r\alpha_1)$, $a_2 = r\alpha_1/(\alpha_2 + r\alpha_1)$. As in paragraph 1, we can show that a mixture of any two diluted mixtures has its corresponding point upon the segment determined by the diluted mixtures, and the position of the point for a given mixture can be similarly obtained.

3. Mixtures of Three Substances.[1] If mixtures of three substances P_1, P_2, P_3 are to be considered, we take an equilateral triangle whose sides are each of unit length and whose vertices are P_1, P_2, P_3. A mixture of a_1 parts[2] of P_1, a_2 parts of P_2, and a_3 parts of P_3 will be represented by a point Q within the triangle, which is determined as follows: upon the segment P_1P_2 determine a point Q_3 such that $P_2Q_3 = a_1$. Similarly upon the segment P_2P_3 determine the point Q_1 such that $P_3Q_1 = a_2$; draw a line through Q_3 parallel to P_2P_3 and a line through Q_1 parallel to P_3P_1; these two lines meet in Q, the required point. We denote by (a_1, a_2, a_3) the *coördinates* of Q, which lies within the triangle when the coördinates are all positive.



Through Q draw a line parallel to P_1P_2 meeting P_1P_3 in Q_2 and P_3P_2 in M, and a line through Q_1 parallel to P_1P_2 meeting P_1P_3 at N. Then $P_3Q_1 = P_3N$; $P_1Q_2 = P_1Q_2$; $P_2Q_3 = QM = QQ_1 = Q_2N$ so that $P_3Q_1 + P_1Q_2 + P_2Q_3 = P_1P_3 = 1$. Since $P_3Q_1 = a_2$, $P_2Q_3 = a_1$, $a_1 + a_2 + a_3 = 1$, it follows that $P_1Q_2 = a_3$, so that the same point Q would be obtained by using *any* two of the three values a_1, a_2, a_3. This justifies calling (a_1, a_2, a_3) the coördinates of Q. In the sequel we shall call Q_1, Q_2, Q_3 the *projections* of Q upon the sides of the triangle $P_1P_2P_3$ in the proper order. The vertices of the triangle $P_1P_2P_3$ correspond to the substances P_1, P_2 and P_3; the points on each of the three sides correspond to mixtures of two substances only, and the points within the triangle to mixtures of all three substances.

To determine the point $Q(q_1, q_2, q_3)$ corresponding to a mixture $Q(q_1, q_2, q_3)$ we proceed as already indicated;[3] to determine a mixture corresponding to point Q within the triangle we find the projections of Q, namely Q_1, Q_2, Q_3, upon the three sides of the triangle, then q_1, q_2, q_3 are obtained from the equations $P_2Q_3 = q_1$, $P_3Q_1 = q_2$, $P_1Q_2 = q_3$, and give the required parts.

4. The Use of a General Triangle for Mixtures of Three Substances. It is

[1] Methods of setting up a correspondence between the points of a triangle and mixtures of three substances have been given by Gibbs, *Transactions, Connecticut Academy*, vol. 3 (1876), p. 176, and by Roozeboom, *Zeitschrift für physikalische Chemie*, vol. 15 (1894), p. 145. See also Bancroft, *The Phase Rule*. The method used here is capable of easy generalization (see paragraph 4) and leads to some results (see paragraphs 5–8) which the writer believes to be new. The coördinate system set up is the one usually known as Areal; see C. A. Scott, *Modern Analytic Geometry*, and Askwith, *Analytic Geometry of the Conic Sections*, chapter XIV. A recent treatment of the applications of this method can be found in Haskell, *How to Make and Use Graphic Charts*, pp. 30–36.

[2] In the remainder of this paper by "parts" will be meant percentages of the whole mixture.

[3] In this representation of mixtures of three substances by points within an equilateral triangle, one may start in either of two ways. The intersection of the line through Q parallel to P_2P_3 with P_1P_3 instead of P_1P_2 might have been taken. In either case the three lines through Q must be taken parallel to the three sides of the triangle in a cyclic order, and the distance to the projection on each of the three sides must be measured from the vertex which is on the side parallel to the projecting line.

evident that the equilateral triangle with unit side, which has been used in the previous paragraph, might be replaced by an equilateral triangle with side equal to ρ, provided the distances measured upon the sides were ρa_1, ρa_2, ρa_3 instead of a_1, a_2, a_3 respectively, as this would merely magnify (if $\rho > 1$) the original configuration. In such a case one measures a_1 along the side of the triangle, regarding the side of the triangle as of unit length.

If one wishes to consider a mixture of three substances, each of which is a mixture of three fundamental substances, difficulty is encountered because the three points which correspond to the three mixtures will not in general form an equilateral triangle; so that we need a method for determining the point corresponding to a mixture of three substances when the corresponding triangle is arbitrary (except that the three vertices must not be collinear). This method follows: Let $P_1P_2P_3$ be a triangle with sides $P_1P_2 = \lambda$, $P_2P_3 = \mu$, $P_3P_1 = \nu$. The point Q corresponding to a mixture Q whose percentages of P_1, P_2, P_3 are a_1, a_2, a_3 respectively is determined as follows. On the segment P_1P_2 determine a point Q_3 such that $P_2Q_3 = \lambda a_1$, upon P_2P_3 determine Q_1 such that $P_3Q_1 = \mu a_2$, draw a line through Q_3 parallel to P_2P_3, and a line through Q_1 parallel to P_3P_1; these two lines meet in Q, the required point.

In order to justify this method, one needs to show that if a line through Q parallel to P_1P_2 meets P_1P_3 in Q_2, then $P_1Q_2 = \nu a_3$, for then the point Q is the same whichever pair of the coördinates (a_1, a_2, a_3) is taken. To show that this is true, prolong QQ_2 to meet P_2P_3 at M, and draw a line parallel to P_1P_2 through Q_1 meeting P_3P_1 at N. Then using similar triangles

$$\lambda a_1 = P_2Q_3 = QM = \frac{\lambda}{\nu} QQ_1 = \frac{\lambda}{\nu} Q_2N,$$

whence $Q_2N = \nu a_1$,

$$\mu a_2 = P_3Q_1 = \frac{\mu}{\nu} P_3N,$$

whence $P_3N = \nu a_2$, so that $P_1Q_2 = \nu - \nu a_1 - \nu a_2 = \nu a_3$.

The method in this case is merely to measure from the proper vertices the distances a_1, a_2, a_3 upon the three sides regarding in each case the length of the side as unity, and to draw lines through these points parallel to the sides of the triangle in the proper order, the three lines so drawn meeting in the required point.

5. Mixtures of Mixtures of Three Substances. Let A, B, C denote mixtures of P_1, P_2, P_3 and also denote the points corresponding to those mixtures; let the coördinates of A, B, C be $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$. Let M denote a mixture of A, B, C with coördinates $M(\mu_1, \mu_2, \mu_3)$ with respect to the triangle ABC (see Figure), the point M corresponding to the mixture M being determined with respect to the triangle ABC by the method described in paragraph 4. We wish to show that the point M so determined corresponds to a ratio mixture M of the substances P_1, P_2, P_3 which is made up of μ_1 parts of A, μ_2 parts of B and μ_3 parts of C. We denote the composition of A by the

symbolic equation $A = a_1P_1 + a_2P_2 + a_3P_3$; similarly $B = b_1P_1 + b_2P_2 + b_3P_3$, $C = c_1P_1 + c_2P_2 + c_3P_3$, $M = \mu_1A + \mu_2B + \mu_3C$, whence

$$\begin{aligned} M = {} & (\mu_1a_1 + \mu_2b_1 + \mu_3c_1)P_1 \\ & + (\mu_1a_2 + \mu_2b_2 + \mu_3c_2)P_2 \\ & + (\mu_1a_3 + \mu_2b_3 + \mu_3c_3)P_3. \end{aligned}$$

We wish to show, then, that the coördinates of M with respect to the triangle $P_1P_2P_3$ are $(\mu_1a_1 + \mu_2b_1 + \mu_3c_1,\ \mu_1a_2 + \mu_2b_2 + \mu_3c_2,\ \mu_1a_3 + \mu_2b_3 + \mu_3c_3)$. Let A_3, B_3, C_3, Q_3 be the projections of A, B, C, M upon the side P_1P_2; draw MM_3 parallel to BC and meeting AB at M_3, and let T_3 denote the projection of M_3 on the side P_1P_2. Then $P_2A_3 = a_1 - b_1$, and by similar triangles

$$\frac{A_3B_3}{AB} = \frac{A_3T_3}{AM_3} \qquad \text{or} \qquad \frac{a_1 - b_1}{-\lambda} = \frac{A_3T_3}{\lambda(1 - \mu_1)},$$

whence $A_3T_3 = (1 - \mu_1)(b_1 - a_1)$; also

$$\frac{MM_3}{CB} = \frac{Q_3T_3}{C_3B_3} \qquad \text{or} \qquad \frac{\mu\mu_3}{\mu_3} = \frac{Q_3T_3}{c_1 - b_1},$$

whence $Q_3T_3 = \mu_3(c_1 - b_1)$.

Therefore $P_2Q_3 = P_2A_3 + A_3T_3 + T_3Q_3 = \mu_1a_1 + \mu_2b_1 + \mu_3c_1$, which is the first coördinate of M with respect to $P_1P_2P_3$; a similar method will determine the other two coördinates.

The theorem just proved has several important corollaries:

1. If the points corresponding to three mixtures A, B, C of three substances P_1, P_2, P_3 are plotted and taken as the vertices of a triangle, then a fourth mixture M can be made from the mixtures A, B, C if and only if the point M corresponding to this mixture M lies within or on the boundary of the triangle ABC.

2. If the point M corresponding to a mixture M lies within the triangle corresponding to three mixtures A, B, C, the parts of A, B, C which will give the mixture M may be determined as follows (see Figure): draw the projections of M upon the triangle ABC, viz., M_3, M_1, M_2; then

$$\mu_1 = \frac{BM_3}{BA}, \qquad \mu_2 = \frac{CM_1}{CB}, \qquad \mu_3 = \frac{AM_2}{AC}$$

are the required values.

3. If the point M lies on one side of the triangle ABC, then the mixture M can be made up of two of the three mixtures A, B, C.

4. The various mixtures which result from taking a fixed proportion of A, and varying the proportions of B and C, have their corresponding points upon a line which is parallel to BC.

5. If the points corresponding to three mixtures lie upon a line, then one of the mixtures can be obtained from the other two, and all the mixtures of the three will have their corresponding points on a line.

6. Mixtures of Three Diluted Substances. A treatment of mixtures of four substances would require the use of a tetrahedron in space. However, the previous discussion and results may be extended to include mixtures of three active substances and one inactive substance. If the concentration of each of the substances considered is the same, the extension is immediate; otherwise some additional argument is necessary.

Let A, B, C be three mixtures of the active substances P_1, P_2, P_3 and an inactive substance; let the coördinates [1] of the corresponding points be $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$ and the concentrations be α, β, γ respectively. Let A', B', C' be mixtures equivalent to A, B, C, each having a concentration of .01. Then a mixture M' equivalent to a mixture M made from μ_1, μ_2, μ_3 parts of A, B, C respectively could be made from $100\mu_1\alpha$, $100\mu_2\beta$, $100\mu_3\gamma$ parts of A', B', C' respectively. Hence the coördinates of M', and so of M, are proportional to $\mu_1\alpha$, $\mu_2\beta$, $\mu_3\gamma$. The point M may therefore be found by the method explained in paragraph 4, and will lie within the triangle ABC, or upon some side of it.

Conversely, let the points corresponding to mixtures A, B, C form the vertices of a triangle and let M denote a point within the triangle ABC; then a mixture corresponding to M can be mixed from A, B, C and the relative proportions of each found from the equalities $BM_3/BA = \rho\mu_1\alpha$, etc., where M_3 is the projection of M upon BA, ρ a factor of proportionality, α, β, γ the concentrations of A, B, C and μ_1, μ_2, μ_3 the required proportions.

7. Mixing Fertilizers. The situation studied in paragraph 6 arises when it is desired to mix a given ratio fertilizer from three standard ones, if possible.[2] The method in this case would be to plot the points corresponding to the three standard fertilizers, then to see whether the point corresponding to the desired fertilizer is within or upon the boundary of the triangle formed by the first three. If not, the desired mixture cannot be obtained from the standard ones; if within, the mixture can be obtained, and the proportions found as indicated in paragraph 6. If one imposes the condition $\mu_1 + \mu_2 + \mu_3 = 2000$, the number of pounds of each standard fertilizer required to make a ton of the mixture is obtained. The concentration of the fertilizer must then be obtained from the concentrations of the standard fertilizers and the proportions used. If this for example is .1 when the desired fertilizer should have .2, then twice as much of the mixture should be used as would be used if the strength were .2.

It may also happen that one wishes to form a mixture of two standard fertilizers A and B in such a way as to obtain a fertilizer as nearly equivalent as possible to a given mixture M; [3] the foot of the perpendicular dropped from M upon AB corresponds to the required mixture.

[1] Here we take as the coördinates of a point the values of the percentages taken of the three active substances in a definite order. The sum of the coördinates of a point will equal the concentration of the corresponding mixture. Two equivalent mixtures will have the same corresponding point.

[2] Here P_1, P_2, P_3 are Nitrogen, Potash, Phosphorus, and the inactive substance is usually called filler.

[3] A mixture P will be said to be more nearly equivalent to a mixture Q than a mixture R when the distance PQ is less than the distance QR, the points P, Q and R being plotted as indicated in paragraph 6.

8. Solution of Three Linear Simultaneous Equations by Graphical Analysis. Let three active substances be denoted by P_1, P_2, P_3 and three mixtures of them by $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$. Further let $M(\mu_1, \mu_2, \mu_3)$ denote a mixture made from x_1, x_2, x_3 amounts of A, B, C respectively; then x_1, x_2, x_3 are solutions of the three equations

$$(A) \qquad \begin{cases} x_1 + x_2 + x_3 = \lambda_1 \\ \dfrac{a_1x_1 + b_1x_2 + c_1x_3}{a_3x_1 + b_3x_2 + c_3x_3} = \dfrac{\mu_1}{\mu_3} = \lambda_2 \\ \dfrac{a_2x_1 + b_2x_2 + c_2x_3}{a_3x_1 + b_3x_2 + c_3x_3} = \dfrac{\mu_2}{\mu_3} = \lambda_3 \end{cases}$$

where λ_1, λ_2, λ_3, a_1, $\cdots$ c_3 are positive (or zero except that $\lambda_1 \neq 0$) constants. Conversely, given a set of equations of the form (A), there will correspond a problem in mixtures which can be solved graphically, and so the equations also can be solved graphically.

The methods employed in this paper may be extended to include points with negative coördinates, an extension useful in solving equations but not in making mixtures. Moreover, a set of equations

$$(B) \qquad \begin{cases} \alpha_1x + \beta_1y + \gamma_1z = \delta_1 \\ \alpha_2x + \beta_2y + \gamma_2z = \delta_2 \\ \alpha_3x + \beta_3y + \gamma_3z = \delta_3 \end{cases}$$

where α_3, β_3, γ_3, δ_3 are all different from zero may be put into the form (A) by the substitution $\alpha_3x = u$, $\beta_3y = v$, $\gamma_3z = w$, becoming

$$u + v + w = \delta_3,$$

$$\frac{\frac{\alpha_1}{\alpha_3}u + \frac{\beta_1}{\beta_3}v + \frac{\gamma_1}{\gamma_3}w}{u + v + w} = \frac{\delta_1}{\delta_3},$$

$$\frac{\frac{\alpha_2}{\alpha_3}u + \frac{\beta_2}{\beta_3}v + \frac{\gamma_2}{\gamma_3}w}{u + v + w} = \frac{\delta_2}{\delta_3},$$

and so may be solved except in special cases. These special cases will arise when three of the points $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$, $(\delta_1, \delta_2, \delta_3)$ are collinear. Finally if in set (B) at least one equation actually contains an α, at least one a β, at least one a γ, and at least one a δ, a linear combination of the three equations can be obtained containing no vanishing constant, and this equation can be used in place of

$$\alpha_3x + \beta_3y + \gamma_3z = \delta_3.$$

ON THE CIRCLES OF ANTISIMILITUDE OF THE CIRCLES DETERMINED BY FOUR GIVEN POINTS.

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1. The following discussion is restricted to the finite real plane, except as imaginary elements are specifically mentioned. By definition, the word *circle* is extended to include straight lines. An *inversion* in a proper circle is a transformation by reciprocal radii. An inversion in a straight line is a reflection. A *circle of antisimilitude* of two circles is a circle, with regard to which they are mutually inverse. It is known that two intersecting circles always have two circles of antisimilitude which intersect orthogonally at their common points, and have their centers at the centers of similitude of the given circles. When two given circles do not intersect, one circle of antisimilitude is real and one is imaginary. If an inversion is performed with regard to a center of inversion which is on a circle of antisimilitude, the given circles invert into equal circles. Hence two circles can always be inverted into equal circles; and three given circles can be so inverted, provided that any circle of antisimilitude of any pair meets a circle of antisimilitude of another pair.

2. It is proposed to study the circles of antisimilitude of the four circles determined by four arbitrary points not on a circle. The figure may be simplified by an inversion whose center is one of the given points; the four circles in question then become the sides and circumcircle of a triangle. We shall consider the simplified figure first.

3. Let the vertices of a triangle be A_1, A_2, A_3; its incenter U, its excenters V_1, V_2, V_3. Let P_i be the mid-point of that arc A_jA_k of the circumcircle which is opposite A_i, and Q_i the mid-point of that which contains A_i. By a well-known theorem,[1] P_i is the center of a circle through A_j, A_k, U, V_i; and Q_i is the center of a circle through A_j, A_k, V_j, V_k. We shall designate these circles by p_i, q_i respectively. The line A_jA_k shall be designated as usual by a_i, and the circumcircle by c.

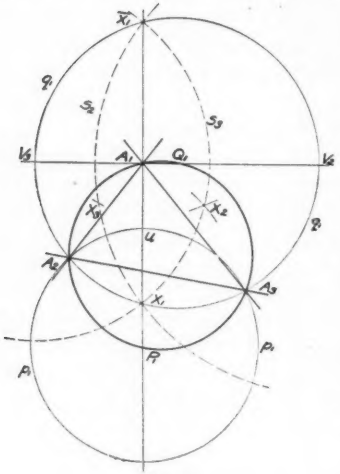

FIG. 1

The circles of antisimilitude of the lines a_j and a_k are the bisectors of the angle A_i. The circles of antisimilitude of the line a_i and the circle c are the circles p_i, q_i. These circles of antisimilitude are concurrent by sixes at the points U, V_1, V_2, V_3. Hence if any one of these points is taken as center of inversion, the inverses of a_1, a_2, a_3, c are equal circles, and the inverses of A_1, A_2, A_3, together with the center of inversion, form an orthocentric system.[2]

[1] For references, see R. C. Archibald, this MONTHLY, *1921*, 229.

[2] R. A. Johnson, this MONTHLY, *1916*, 161; Arnold Emch, *ibid.*, *1916*, 162.

The circle q_i meets the interior bisector of A_i at two points which we will call X_i, $\overline{X}_i$. The circle p_i does not meet the exterior bisector of A_i in real points; we may say for convenience that this line and circle meet in two imaginary points Y_i, $\overline{Y}_i$. These twelve points complete the tally of points of intersection of the twelve circles of antisimilitude.

The six points (X) *lie by fours on three circles* s_1, s_2, s_3 *whose centers are respectively* V_1, V_2, V_3, *and which are mutually orthogonal. Each of them is orthogonal to three angle bisectors and three p- and q-circles.*

For

$$UX_j \cdot U\overline{X}_j = UA_i \cdot UV_i = UX_k \cdot U\overline{X}_k,$$

whence X_j, $\overline{X}_j$, X_k, $\overline{X}_k$ are concyclic. The center of the circle is obviously V_i; there are three such circles, and since

$$\angle V_jX_iV_k = \angle V_jA_iV_k = 90°,$$

any two of them are orthogonal. Obviously also U is their radical center. We designate by s_i the circle through X_j, $\overline{X}_j$, X_k, $\overline{X}_k$.

By an inversion in s_i, the circle $V_iX_j\overline{X}_j$ inverts into the line $X_j\overline{X}_j$; that is, by this inversion A_i, A_j, A_k exchange positions respectively with U, V_k, V_j. It follows that s_i is orthogonal to the circles q_i, p_j, p_k, as well as to those angle bisectors which pass through V_i. Each of the other three bisectors inverts into a p- or q-circle. Thus:

Each circle s_i *is orthogonal to six of the twelve circles of antisimilitude, and the other six are by pairs inverse with regard to it. It is therefore coaxal with each of the three pairs.*

For instance, s_i is coaxal with $V_jA_iV_k$ and p_i, and therefore passes through the imaginary points Y_i, $\overline{Y}_i$ mentioned above. We easily see that an inversion in the center U, with radius given by

$$r^2 = \overline{UA}_i \cdot \overline{UV}_i \quad \text{(a negative constant)},$$

interchanges each A_i with the corresponding V_i, and also each X_i with $\overline{X}_i$. (In terms of real elements only, this transformation is an inversion whose radius is $\sqrt{\overline{A_iU} \cdot \overline{UV}_i}$, followed by a rotation through 180°.) By this inversion, then, each circle s_i is unchanged, and the circle of inversion is therefore their common orthogonal circle. This same inversion moreover interchanges the two circles which intersect at Y_i and $\overline{Y}_i$; hence the six imaginary points $(\overline{Y})$ lie on this imaginary circle, and are its intersections with s_1, s_2, s_3.

The twelve points (X), (Y) *are the total intersection of a set of four mutually orthogonal circles whose centers are* U, V_1, V_2, V_3. *The six imaginary points* (Y) *lie on the imaginary circle whose center is* U.

We may see incidentally that *the four circles each of which passes through three of the four points* U, V_1, V_2, V_3 *are equal; their twelve circles of antisimilitude are precisely the same angle bisectors and p- and q-circles as those of the given four circles.*

4. By an inversion with regard to an arbitrary center, which does not lie on any of the lines or circles of the figure, we are led to the following more general results.

Let four arbitrary points be A_1, A_2, A_3, A_4; denote by c_{ijk} the circle through A_i, A_j, A_k, and designate by t_{ij}, t_{ij}' the circles of antisimilitude of c_{ijk} and c_{ijl}. These twelve circles of antisimilitude are concurrent by sixes at four real points B_1, B_2, B_3, B_4. Further, if the primes are suitably assigned, any circle t_{ij} meets t_{kl}' in two real points (X), and the corresponding pair t_{ij}', t_{kl} meet in two imaginary points (Y). These twelve points are the total intersection of four mutually orthogonal circles (s); on one of these four circles are the six imaginary points (Y), and on each of the others are two imaginary and four real points.

Each circle (s) is orthogonal to six of the circles of antisimilitude (t), and the other six are by pairs mutually inverse with regard to it. With regard to any one of the s-circles, the tetrads $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are mutually inverse; incidentally, these two quadrangles are four times in perspective, namely at the centers S_1, S_2, S_3, S_4 of the s-circles,[3] which latter constitute an orthocentric system. The t-circles are also circles of antisimilitude of the four circles determined by the four points B_1, B_2, B_3, B_4.

If one of the points B_i is taken as center of inversion, the A's invert into an orthocentric system, and the remaining B's into the feet of the altitudes of this system; and vice versa, if an A_i is taken as center of inversion. If a point (X) is taken as center of inversion, each quadrangle inverts into a parallelogram. The two parallelograms are congruent and have common diagonals and center. That is, in the inverted figure $A_iB_jB_kA_l$ and $B_iA_jA_kB_l$ are similar rectangles, similarly placed with regard to a common center.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. SHOULD BOOK REVIEWS BE CENSORED?

By L. E. DICKSON, University of Chicago.

A mathematical manuscript is never accepted by a journal in America until it has been approved by at least one referee. But book reviews appear to escape such a check. This is unfortunate. An error in an original paper reflects only upon the author. But errors in a book review usually do not reflect upon the reviewer as they should, but often do great injustice to the author of the book.

[3] The twelve points (A), (B), (S) are, in other words, the plane projections of a desmic configuration.

Most authors prefer to suffer in silence rather than attempt impracticable redress. Surely book reviews should be submitted to impartial referees prior to publication.

Several years ago the writer found out first hand from those then writing most of the reviews that it was no longer the fashion to write frankly concerning the merits or shortcomings of books, but that one should "damn with faint praise" and carefully conceal from the dear public the reviewer's real opinion of a book. Often what the reviewer said in several pages could have been said in the single sentence: "This is a good book, but I hate like the deuce to admit it."

Unfortunately the American reviewer often does not aim primarily to portray the degree of success attained by a book in the choice of subject matter and in its clarity of presentation, but rather to convince the reader that the reviewer himself is a person who knows a great deal about the subject and how it should have been presented.

If I select a specific case for illustration, I must choose one whose consideration will do no harm to the reviewer, author, or publisher. These requirements are met in the case of the recent favorable review of my *First Course in the Theory of Equations* by Professor Bennett. This review appeared in the MONTHLY in the final number for 1922, published April 9, 1923. The text had appeared a year earlier and was already familiar to nearly all teachers of collegiate mathematics, as shown by the fact that it has had a larger sale than any other mathematical text beyond the calculus. Hence I cannot be accused of attempting here to advertise this text.

The vague remarks comprising the first page of the review imply that the reviewer is dissatisfied with the traditional choice of subject matter in books on the elementary theory of equations, although he admits that this choice is acceptable to most teachers of it. The second page gives a kindly appreciation of the general features of the book. A far more concrete idea of the book was given in the same amount of space in the review by Robert d'Adhémar in the *Bulletin des Sciences Mathématiques* for 1922.

Since the review by Bennett is exceptionally favorable to the book, why do I call attention to it now? Solely because the collection on the final page of ten "minor infelicities" contains instances of amazing carelessness on the part of the reviewer.

In a proof (p. 18) by mathematical induction (a process explained very fully and simply on p. 5), I stated that "if we multiply each member of identity (9) by $x - \alpha_{n+1}$, it is not much trouble to verify that the resulting identity can be derived from (9) by changing n into $n + 1$, so that (9) is proved true by mathematical induction." The reviewer complained that "in connection with mathematical induction the author speaks of 'changing n into $n + 1$,'" with the implication that the author had committed the error often made by a novice in induction.

I am at loss to know why he condemns my phrase "A point on the graph at which the tangent is both horizontal and an ordinary tangent is a bend point . . .," since it is both mathematically exact and expressed clearly.

"The discriminant, after being defined in general on page 47, is redefined for the quartic on page 51." In the earlier case the definition had been applied to write out the discriminant of the cubic only. In the new case there was need of the explicit expression for the discriminant of the quartic and it was justified by repeating the definition. Clearness and continuity of thought should not be a crime.

On page 21 I said that "any number which exceeds all real roots of a real equation is called an *upper limit to the real roots.*" The reviewer complained that the italicized phrase is defined as a single concept without any inquiry as to the meaning of the word "limit" or the sense of "upper." Why did he fail to criticize for similar reasons my definition of "integral rational function" as synonymous with "polynomial"? When one desires to clarify the distinction between the right-hand and left-hand pages of an open book, does he need to enter upon the anatomical properties of the hand as distinguished from the foot? The many teachers who are using this elementary text will approve of my abstaining from entering upon the delicacies inherent in the notion of a limit, and my sensible definition of the above entire phrase as a compound name.

Since I spoke of the trigonometric form (in polar coördinates) of $a + bi$, the reviewer says I should have given also a separate name to $a + bi$. But I had called $a + bi$ a complex number and now found need to speak of its *trigonometric form.*

"Immediately after emphasizing the fact that a complex number has many amplitudes differing by integral multiples of 360°, the term 'the amplitude' is used without a hint as to which one is 'the amplitude.' The facts are that, on the last two lines of page 3, I said, 'For the amplitude we may select, instead of θ, any of the angles $\theta \pm 360°$, $\theta \pm 720°$, etc.,' while in the succeeding fifteen lines I used the term 'an amplitude' twice and spoke of 'the indicated amplitudes' (i.e., amplitudes 120° and 240° indicated in the figure), but never employed the imputed term 'the amplitude.'"

"In the exercises on page 74 the letters must denote only real quantities although no such warning is given." But the theorem under which these exercises fall, as well as the heading of the chapter, explicitly stated that we are dealing with equations with real coefficients. Moreover, all of the literal coefficients which appear in the exercises are explicitly squares, with the evident object of implying that these coefficients are not merely real, but positive.

It happens that I am now sending eight minor corrections to the plates in anticipation of a new printing; a careful examination of the above "infelicities" convinced me that no correction is needed on account of them.

Since the reviewer, himself an expert, has gone patiently through the text seeking faults, with no better success than the mentioned groundless infelicities, I conclude that the text is remarkably free from faults, thanks partly to the minute criticisms of the manuscript and proof-sheets by the specialists cited in the preface. For the trouble of writing also this second elementary text on the theory of equations, my real reward has been the knowledge that it is now

yearly inspiring thousands of students with a genuine interest and affection for mathematics.

II. Infinite and Imaginary Elements in Algebra and Geometry.

By Tomlinson Fort, University of Alabama.

Since the things advocated by Professor R. M. Winger in his article bearing the above title and appearing in a recent number of the Monthly (*1922*, 290–296) are at variance with the general practice followed by me in my teaching of freshman and sophomore classes, I am prompted to comment on what he says.

In my opinion one of the duties of the teacher of elementary mathematics is to attempt to rid the subject of mysticism and to make it appear at all times in conformity with the student's experience. Like most teachers I welcome elementary uses for imaginary numbers but can not believe the first course in analytic geometry the place for them. To the freshman, points and lines are more or less the pictures that he sees and are not ideals. His analytic geometry is truly a geometry of reals and for the instructor to confine his teaching to the "real domain" seems to me sound mathematically and to rid the subject of that element of mystery which I do not believe it possible to eliminate when trying to teach geometry with imaginary elements to such immature students. I never feel that I am telling the freshman a half truth when I tell him that (0, 0) is the only point satisfying $x^2 + y^2 = 0$ or that $x^2 + y^2 + 1 = 0$ has no locus. I think that imaginary elements should be introduced at a later time when the whole subject can be subjected to a more critical examination. The same is true of infinite regions of the plane. To bring a class of freshmen to understand and remember their ideal nature seems to me out of the question. I simply never mention the subject: Parallel lines do not meet. The parabola is an open curve and division by zero is never permitted.

It is with a kind of horror that I read where the author advocates the postulation of "*The Number Infinity*" defined by $k/0 = \infty$, $k \neq 0$. How I fight division by zero in my classes! How many times have I told my students that 90° has no tangent and have forbidden this same infinity! To describe the behavior of the roots of an equation,

$$a_0x^n + \cdots + a_n = 0; \qquad a_0 \neq 0,$$

when $a_0, \cdots, a_{r-1}$ approach zero seems to me sound teaching, but to speak of the equation as having r infinite roots when $a_0 = \cdots = a_{r-1} = 0$ and the equation is only of the $(n - r)$th degree, as the author advocates, is to me very bad. It is just that from which I thought American mathematics was growing. The author states that what he advocates is in accordance with "*sound* European tradition." If I understand him correctly, I am sure in particular that his infinity was not in use in the teaching at Göttingen in 1912 or in Paris in 1913 when I was a student at those universities. Also in my opinion he will find that it has been eliminated from the better teaching in England. With reference to infinity,

I think the best policy in elementary teaching is to explain carefully the various unending processes that arise and in particular to make it clear that there is no largest real number. In more advanced courses where infinite elements are introduced care should be taken to explain their ideal character and to show just why it is wise to postulate them in exactly the form that it is done.

III. A Theorem Concerning the Concurrency of Four Planes.

By E. L. Rees, University of Kentucky.

In a previous number of this Monthly (*1920*, 165) the author gave a theorem on the concurrency of three lines by means of which certain theorems of geometry could be proved in a very brief and simple way. In the present note a similar theorem is given, relating to the concurrency of four planes, with applications to illustrate its use.

Let $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$ be the position vectors of three points determining a plane, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{B}_3$ those of a second plane, and so on. *A necessary and sufficient condition that four planes be concurrent is that* $\Sigma a\boldsymbol{A} = \Sigma b\boldsymbol{B} = \Sigma c\boldsymbol{C} = \Sigma d\boldsymbol{D}$, *where* $a_1, a_2, a_3, \cdots, d_1, d_2, d_3$ *are some quantities, not all zero, satisfying the equations* $\Sigma a = \Sigma b = \Sigma c = \Sigma d$.

The proof is omitted, being almost identical with that given for the theorem referred to.

If $\Sigma a\boldsymbol{A} = \cdots = 0$, and $\Sigma a = \cdots \neq 0$, the planes intersect at the origin.

If $\Sigma a\boldsymbol{A} = \cdots \neq 0$, and $\Sigma a = \cdots = 0$, the planes intersect at infinity.

If $\Sigma a\boldsymbol{A} = \cdots = 0$, and $\Sigma a = \cdots = 0$, the vectors of each sum are terminocollinear and the planes are not determined. We exclude this case.

As applications we prove the following theorems.

The six planes determined by the edges of a tetrahedron and points taken on the opposite edges are concurrent if the ratios in which the points divide the opposite edges are such that the product for each face is unity, and conversely.

Proof: Let the vertices of the tetrahedron be A, B, C, D, and let the division points E, F, G, H, I, J be taken in accordance with the equations below. The hypothesis of the theorem is satisfied in the most general way in the following equations.

$$\begin{array}{lll} (a+b)\boldsymbol{E} = a\boldsymbol{A} + b\boldsymbol{B}, & (a+d)\boldsymbol{G} = a\boldsymbol{A} + d\boldsymbol{D}, & (b+d)\boldsymbol{I} = b\boldsymbol{B} + d\boldsymbol{D}, \\ (a+c)\boldsymbol{F} = a\boldsymbol{A} + c\boldsymbol{C}, & (b+c)\boldsymbol{H} = b\boldsymbol{B} + c\boldsymbol{C}, & (c+d)\boldsymbol{J} = c\boldsymbol{C} + d\boldsymbol{D}, \end{array}$$

where a, b, c, d are any scalar quantities. Add $c\boldsymbol{C} + d\boldsymbol{D}$ to each member of the first equation, $b\boldsymbol{B} + d\boldsymbol{D}$ to each member of the second, etc. It results that

$$\begin{aligned} &(a+b)\boldsymbol{E} + c\boldsymbol{C} + d\boldsymbol{D} = (a+c)\boldsymbol{F} + b\boldsymbol{B} + d\boldsymbol{D} = (a+d)\boldsymbol{G} + b\boldsymbol{B} + c\boldsymbol{C} \\ &= (b+c)\boldsymbol{H} + a\boldsymbol{A} + d\boldsymbol{D} = (b+d)\boldsymbol{I} + a\boldsymbol{A} + c\boldsymbol{C} = (c+d)\boldsymbol{J} + a\boldsymbol{A} + b\boldsymbol{B}, \end{aligned}$$

the scalar coefficients of any four of which satisfy the conditions of the theorem. Hence the proof is complete.

Corollary: The four lines joining the vertices of a tetrahedron with the

points of intersection of the lines which join the vertices of the opposite faces with the division points of the sides of these faces are concurrent.

This corollary may also be proved independently by the theorem in the article referred to as follows.

Let A', B', C', D' represent the points on the faces opposite A, B, C, D respectively. We easily find

$$(b+c+d)\mathbf{A}' = b\mathbf{B} + c\mathbf{C} + d\mathbf{D}, \qquad (a+b+d)\mathbf{C}' = a\mathbf{A} + b\mathbf{B} + d\mathbf{D},$$
$$(a+c+d)\mathbf{B}' = a\mathbf{A} + c\mathbf{C} + d\mathbf{D}, \qquad (a+b+c)\mathbf{D}' = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}.$$

Adding $a\mathbf{A}$ to each member of the first equation, $b\mathbf{B}$ to each member of the second, etc., we get as before

$$(b+c+d)\mathbf{A}' + a\mathbf{A} = (a+c+d)\mathbf{B}' + b\mathbf{B} = (a+b+d)\mathbf{C}' + c\mathbf{C} = (a+b+c)\mathbf{D}' + d\mathbf{D}.$$

Since the sums of the scalar coefficients of the different members are equal the corollary is proved. The point of intersection of these lines is given by

$$\frac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + d\mathbf{D}}{a+b+c+d}.$$

If the points of division are the midpoints of the edges we then get the center of gravity of the tetrahedron which is given by $\frac{1}{4}(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})$. The equations also show that this point divides each of the lines joining a vertex with the center of gravity of the opposite face in the ratio 3 : 1.

Conversely, let b/a, c/b, a/d be the ratios in which AB, BC, and DA are divided by three of the given planes; then, as shown above, the points which divide the edges CA, BD, and DC in the ratios a/c, d/b, c/d respectively determine with the opposite edges planes concurrent with the other three. These planes are therefore among those given. The ratios satisfy the conditions of the theorem, hence the converse is proved.

Very similar to the proof just given is that of the following theorem.

The planes determined by the sides of a space quadrilateral and points taken on the opposite sides are concurrent if the product of the ratios in which these sides are divided is unity.

As a last application we prove the following.

A plane cuts each of the six edges of a tetrahedron; another point is taken on each edge, so as to cut it harmonically; prove that the six planes through these latter points and the opposite edges of the tetrahedron intersect in one point. (From Frost's *Solid Geometry*.)

In view of the theorem of Menelaus (the vector proof of which is quite simple) the hypothesis of our theorem is contained in the assumption that the edges AB, BC, CA, BD, DA, CD are divided respectively in the ratios b/a, c/b, $-a/c$, d/b, $-a/d$, $-d/c$ which of course depend on the position of the cutting plane with reference to the tetrahedron.

Denoting by E, F, G, H, I, J the points taken so as to cut the edges harmonically, we have

$$(a-b)\boldsymbol{E} = a\boldsymbol{A} - b\boldsymbol{B}, \quad (-b+c)\boldsymbol{F} = -b\boldsymbol{B} + c\boldsymbol{C}, \quad (c+a)\boldsymbol{G} = c\boldsymbol{C} + a\boldsymbol{A},$$
$$(-b+d)\boldsymbol{H} = -b\boldsymbol{B} + d\boldsymbol{D}, \quad (d+a)\boldsymbol{I} = d\boldsymbol{D} + a\boldsymbol{A}, \quad (d+c)\boldsymbol{J} = d\boldsymbol{D} + c\boldsymbol{C}.$$

Add the terms necessary in each equation to make the right-hand member equal to $a\boldsymbol{A} - b\boldsymbol{B} + c\boldsymbol{C} + d\boldsymbol{D}$. We then have

$$(a-b)\boldsymbol{E} + c\boldsymbol{C} + d\boldsymbol{D} = (-b+c)\boldsymbol{F} + a\boldsymbol{A} + d\boldsymbol{D} = (c+a)\boldsymbol{G} - b\boldsymbol{B} + d\boldsymbol{D}$$
$$= (-b+d)\boldsymbol{H} + a\boldsymbol{A} + c\boldsymbol{C} = (a+d)\boldsymbol{I} - b\boldsymbol{B} + c\boldsymbol{C} = (d+c)\boldsymbol{J} - b\boldsymbol{B} + a\boldsymbol{A},$$

the scalar coefficients of which satisfy the conditions of the theorem, which completes the proof.

The converse of this theorem may be proved by an argument similar to that given above.

IV. On the Evaluation of an Elliptic Integral.

By Rev. M. F. Egan, University College, Dublin.

The following simple method gives, within narrow limits, the approximate value of the complete elliptic integral

$$K = \int_0^{\pi/2} d\varphi/(1 - k^2 \sin^2 \varphi)^{1/2}$$

when k is nearly equal to unity.

Putting $\sin \varphi = \tanh u$, we get

$$K = \int_0^{\infty} du/(1 + k'^2 \sinh^2 u)^{1/2}.$$

Since $k'^2 (= 1 - k^2)$ is small, the term $k'^2 \sinh^2 u$ is of importance only for large values of u, and for these $\sinh u$ is nearly equal to $\frac{1}{2}e^u$. We are therefore led to consider the integral

$$K_0 = \int_0^{\infty} du/(1 + \tfrac{1}{4}k'^2 e^{2u})^{1/2}.$$

Putting $v = e^{-u}$, we get

$$K_0 = \int_0^1 dv/(v^2 + \tfrac{1}{4}k'^2)^{1/2} = \sinh^{-1}(2/k')$$
$$= \log_e (4/k') + k'^2/16$$

to the order k'^2. Since $\sinh u < \frac{1}{2}e^u$, $K > K_0$. On the other hand

$$\frac{1 + \frac{1}{4}k'^2 e^{2u}}{1 + k'^2 \sinh^2 u} = 1 + \frac{k'^2(1 - \frac{1}{2}e^{-2u})}{2(1 + k'^2 \sinh^2 u)}$$
$$< 1 + \tfrac{1}{2}k'^2 < (1 + \tfrac{1}{4}k'^2)^2,$$

so that $K < (1 + \frac{1}{4}k'^2)K_0$. Therefore

$$K_0 < K < (1 + \tfrac{1}{4}k'^2)K_0.$$

RECENT PUBLICATIONS.

REVIEWS.

History of the Theory of Numbers. By L. E. DICKSON. Volume III.[1] Quadratic and Higher Forms. With a Chapter on the Class Number, by G. H. CRESSE. Carnegie Institution, Washington, D. C. March, 1923. v + 313 pages.

The astonishing rapidity with which Professor Dickson has made the manuscript ready for the printer and has seen through the press the first three volumes (1919, 1920, 1923) of his monumental history of the theory of numbers must be a cause for admiration on the part of every one who has witnessed the performance; and the surprise only increases when the reader finds that each volume is done perhaps even more skillfully than the preceding one. In the preface to the third volume we are told that it has been completed promptly owing to the favorable reception accorded to the first two volumes. The preface to the second volume closed with a sentence referring to the third as the concluding volume; but in the third now before us we have (on page 3) a reference forward to volume IV. The reviewer ventures to predict that the favorable reception of the third volume will give the author still more reason for proceeding promptly with the fourth if his astonishing supply of energy is holding out well enough to leave him still susceptible to such influence.

The present reviewer is not so well acquainted with the subject matter of this volume as he is with that of the two preceding; and consequently his judgment about it is entitled to less confidence, especially with reference to the question of completeness; but from what he knows of this matter for volumes I and II and from the fact that the third volume deals with matter which by its nature is more readily accessible to a full treatment and from the evident care of the author and those who have assisted with the perfection of the manuscript and the proofsheets, he feels confident that little will be found missing, so little in fact that it becomes a matter of importance that every reader respond to the author's request by informing him of every omission or error of any sort that may be noticed. The work comes so near to being entirely complete that it is a matter of importance and common interest to supply to it whatever may yet be lacking. The subject-matter of the volume is such that its publication should give a great impetus to the development of the important arithmetic theory of forms. By a "form" is meant a homogeneous polynomial such as $f = ax^2 + bxy + cy^2$, all of whose terms are of the same total degree in x and y, the constant coefficients a, b, c being integers in the greater part of the theory. The arithmetical theory of forms has an important application to the problem of finding all the ways of expressing a given number m in a given form f, that is, of finding all the integral solutions x, y of the equation $ax^2 + bxy + cy^2 = m$. This problem is best attacked generally by considering with f all the equivalent forms g which can be derived from f by applying to the variables linear substitu-

[1] For reviews of Vols. I and II, see this MONTHLY, *1919*, 396; *1921*, 72.

tions with integral coefficients of determinant unity. It is by a simultaneous consideration of all these forms g that one is able to solve completely the equation $f = m$. It is this type of question that is treated in volume III. The results have obvious applications to many of the problems discussed in volume II and to a few of those in volume I. From these remarks it will be seen that volume III deals mainly with general theories rather than with special problems and special theorems. The investigations in question are largely those of leading experts and deal with some of the more advanced parts of the theory of numbers. All previous reports and treatises on forms are now very inadequate owing to the large number of important recent papers on the subject. These are perhaps some of the reasons which cause the author to speak of this volume as "doubtless the most important one of the series."

Of each of the first two volumes the author gave a summary in the preface, this being possible partly on account of the elementary nature of much of the subject-matter and the fact that many of the results were embodied in definite theorems which could be expressed without the use of difficult technical terms. But, in the present volume, "it is a question not primarily of explicit results, but chiefly of general methods of attacking whole classes of problems, the methods being often quite intricate and involving extensive technical terminology." For this reason the preface does not contain a summary of the contents of the volume; but to each of the longer chapters an appropriate introduction and summary are attached.

For the purpose of describing briefly its contents it is convenient to divide the volume (perhaps mechanically rather than logically) into three nearly equal parts, the first two of which deal with binary quadratic forms. One of these consists of the single chapter VI on class number; another, of the remaining chapters I to VIII, all on binary quadratic forms; and the last, of the further chapters IX to XIX, dealing with ternary and quaternary quadratic forms, quadratic forms in n variables, cubic forms, forms of degree greater than three, Hermitian forms, bilinear forms and related matters, representation by polynomials modulo p, and the congruencial theory of forms.

The part first named (chapter VI, pages 92–197) is by G. H. Cresse who devoted five years to the preparation of this report on the difficult subject of the number of classes of binary quadratic forms with integral coefficients, difficult partly on account of the fact that it involves many branches of pure mathematics. It was written as a thesis for the doctorate at Chicago. L. J. Mordell read critically the manuscript of this chapter and also the proofsheets, comparing the whole with the original papers and making numerous suggestions for its improvement. Much of it was examined also by E. T. Bell. The completeness and accuracy of this important chapter are thus made certain by a collaboration of effort in revising and checking. A similar, if less extensive, collaboration marks the preparation of all parts of the book, the proofsheets of each chapter having been read carefully by at least one specialist in the subject of the chapter.

Chapters II, III, IV, V, VII, VIII deal respectively with the following topics

in the theory of binary quadratic forms: explicit values of x, y in $x^2 + \Delta y^2 = g$; composition of binary quadratic forms; orders and genera, their composition; irregular determinants; binary quadratic forms whose coefficients are complex integers or integers of a field; number of classes of binary quadratic forms with complex integral coefficients. The first chapter (pages 1–54), entitled "Reduction and equivalence of binary quadratic forms, representation of integers," contains a brief introduction and explanation of symbols and a treatment of the topics in the theory of binary quadratic forms not belonging to chapters II to VIII. The nature of the contents is made clear by a brief summary.

The subject-matter of the last third of the volume has already been indicated by a summary of the principal chapter headings. It remains to add a few remarks concerning certain of the chapters. The results recorded in some of these are obviously fragmentary; and this suggests the desirability of a systematic analysis of their subject-matter with the purpose of carrying out further investigations so as to put the whole into a more satisfying form. Thus chapter XIII on cubic forms in three or more variables (together with the corresponding pages 594–595 of volume II) will bring to the reader's attention certain problems which it is desirable to have treated more fully, problems where certain of the methods of attack seem to lie rather close to hand. It may be that this topic is accessible to the amateur who has considerable initiative; it is certainly accessible to a good graduate student interested in the theory of numbers. If the matter of this chapter were brought to a fairly satisfactory state it would probably lead the way to a further treatment of certain related forms of degree higher than 3 and to the association of the results with further relevant matters in volume II.

The last chapter, on the congruencial theory of forms, deals with some of the most recent contributions to the theory of numbers. It is due primarily to Dickson and his students. It is a safe prediction that this subject will receive further effective attention in the near future.

It is pleasant to give credit to those to whom it is due when such an extensive and valuable work is put into our hands. First of all, of course, this goes to Dickson himself. Nor must one forget the work of Cresse in the long chapter VI. A significant service has also been rendered by those who have assisted in perfecting the manuscript or in the critical reading of the proofsheets; and to these Dickson has referred in his prefaces in a generous way. The work of those who help with corrections and additions to be incorporated into the promised supplement may also come to deserve our generous appreciation. And we should also remember gratefully the encouragement which has come from the attitude of the Carnegie Institution toward the printing of such an extensive work. Writing in November, 1918, at the close of the preface to volume I, Dickson said, "Finally, this laborious project would doubtless have been abandoned soon after its inception seven years ago had not President Woodward [of the Carnegie Institution] approved it so spontaneously, urged its completion with the greatest thoroughness, and given continued encouragement." This should be a constant

reminder to us of the need of ample means of publishing comprehensive scientific works when persons of ability and suitable preparation are willing to undertake the arduous and unremunerative task of preparing them.

While all this work is fresh in his mind, Professor Dickson, it would seem, is peculiarly well-fitted to prepare a concise outline history of the theory of numbers, indicating the major historical stages in the development of the theory and showing the growth and progress of ideas as the centuries have passed. If he should even venture into the uncertain domain of prophecy and indicate the probable direction of further progress, his predictions would be a matter of interest both now and hereafter. When a master, with the work of the past well in mind, tries to see the trend of the future, his judgment will be a matter of interest whether or not the direction of progress turns out to be such as he anticipates. It may even throw some light on the difficult question as to the way in which new discoveries arise.

The extraordinarily varied development of modern mathematics has led thoughtful persons to inquire what is to be done to prevent the whole discipline from falling apart into chaos. Two possible solutions to the difficulty are thinkable. We may find new and deep-lying principles unifying great bodies of doctrine: this is the ideal solution, but it is not one to be had simply by asking for it; we have to await the discovery or invention or creation of the unifying conceptions. The other (partial) solution is that afforded by full and complete summaries of existing knowledge in certain chapters of science. This will also be needed even if new and deep-lying conceptions arise to unify our knowledge. Indeed such complete summaries as those of Dickson will be of increasing importance whatever direction the development of science may take.

But the needs cannot all be met by these summaries alone. We require also extended and comprehensive expositions of these subjects. This is the more important in the case of disciplines which are somewhat fragmentary in significant parts, especially if their character in this respect leads (as it often does) to the production of a scattered literature containing many fragments. The great increase in the accessibility of the theory of numbers due to Dickson's extensive work whets our appetite for more, especially as the existence of his work seems to render feasible a concerted effort to put the whole of the present theory of numbers into a connected and complete expository form. The suggested task is a great one—and it requires the coöperation of numerous workers. But what better time is there to suggest the undertaking of a great enterprise than that time which witnesses the bringing well along toward completion of another great enterprise which does much to make the new one feasible? If we cannot enter upon the whole of the extended labor, may we not undertake the performance of certain parts of it—those parts which are most needed and give promise of the largest usefulness for the greatest period of time?

R. D. CARMICHAEL.

Higher Geometry. By F. S. Woods. Boston, Ginn & Co., 1922. 16mo. x + 423 pages. Price $5.00.

The sententious Lord Bacon has enriched the world with the statement that "Some books are to be tasted, others to be swallowed, and some few to be chewed and digested." In approaching a new book one naturally asks which of these methods of treatment is the suitable one. The question is peculiarly hard in the present case, for I must confess that my first impression is not of a book at all, but of a moving picture exhibition, where the greatest imaginable variety of geometrical theorems and principles have been flashed upon the screen. The classical examples for such treatment of mathematics are to be found in Klein's lithographed lectures, particularly those of 1893 dealing with "Hoehere Geometrie," the most stimulating, though elusive, introduction to the subject ever written.

The present volume has some of the same characteristics. The author's purpose, as announced in the preface, is "To present some of the general concepts and methods which are necessary for advanced work in algebraic geometry (as distinguished from differential geometry) but which are not now accessible to the student in any one volume, and to bridge the gap between the usual text in analytic geometry and treatises or articles on special topics."

He wishes, then, to build a bridge. I think that he certainly does so. A bridge is built, a very good bridge, only one thing seems lacking, an approach on the nearer side. At the present moment there is a strong tendency in American colleges to cut down the time assigned to analytic geometry to a minimum, in order to introduce the calculus as early and as completely as possible. I am far from disapproving this tendency, but I believe that the pious hope that the student will later go back to the topics in analytic geometry which were formerly required, and are now omitted, is but seldom realized in practice. A student who has done, not all that he might do, nor all that the writer fondly hopes he will do, in the usual analytic geometry text-book, but all that is absolutely required of him before he goes ahead with the calculus, has not, I fear, either the algebraic skill, or the geometric imagination necessary to grasp the full meaning of Professor Woods's stimulating work. Let us run through the pages hastily and see whether this pessimistic judgment is well founded.

The first chapter is of a general abstract character, dealing with such topics as the principle of duality, the use of imaginaries, infinities, transformations and groups; fundamental ideas in modern geometry, in fact, the most fundamental. The idea of group is not, I think, sufficiently stressed, at least that is the impression I get. I think also that in practice a skillful teacher will be needed to elaborate the meaning of the chapter to a class.

Chapters II and III deal with the binary domain, geometry on a single line. This topic is not particularly stimulating to a class, but could not well have been omitted from the present work which proceeds methodically through one dimension, two dimensions, three dimensions, four dimensions, n-dimensions. I wish that the author would not speak (p. 17) of *the* anharmonic ratio of four

elements; we have either *an* anharmonic ratio of four elements, or *the* anharmonic ratio of four ordered elements.

The real substance of the work is first met in Chapter IV, dealing with point and line coördinates in the plane. The usual cartesian coördinates are replaced by homogeneous values $x : y : t$, the plane is extended to be a perfect continuum by the adjunction of the line at infinity, whose circular points are used with good effect. Conic sections are treated rather "stepmotherly"; they only get two pages. The fact is that they appear at length in the next chapter under the more pompous title of "curves of the second order in trilinear coördinates." No connection is established between the two. That is left to the reader's imagination. Trilinear coördinates are introduced at first by a geometric treatment, their expression as linear homogeneous functions of the homogeneous cartesian coördinates is not taken up till later. The chapter ends with a short discussion of theorems about lines and linear transformations, and with the general definition of curves and envelopes. One or two inaccuracies should be noted in passing. There is a good deal of obscurity as to the meaning of the words "real," "imaginary," "complex." Thus we read (p. 28), "If one or more of the coefficients (in the homogeneous equation of a straight line) are complex, the line is said to be *imaginary.*" It is interesting to note that "*an imaginary straight line has one and only one real point.*" It would appear from this as if the author did not wish to include real quantities as a special case of complex ones, and as if he had entirely forgotten about the case of a line with imaginary coefficients whose ratios are all real. The Taylor expansion for

$$f(y_1 + lz_1, y_2 + lz_2, y_3 + lz_3)$$

is used to find the conditions for a singular point on a curve, but not for finding the equation of the tangent at a non-singular one, a curious omission.

Chapters V and VI are given to curves of the second order and the second class, and to linear transformations. The treatment of these subjects is pretty well standardized in books on projective geometry, and the present work offers little new. I find an original note in the discussion of the intersections of two conics which is based on a study of the collineation obtained by successive polarization in the two. This chapter, like all the others, is enriched with abundant examples for the student to work out; I find those on p. 64 particularly good. Page 70 starts with the unfortunate statement that if the determinant of the coefficient of three non-homogeneous linear equations in three variables be zero, there can be no solutions. On p. 83 we find a proof that the characteristic equation is the same in both point and line coördinates, a mistake. The writer considers a certain transformation in point coördinates and its inverse in line coördinates.

Chapter VII shows the first departure from traditional topics by taking up projective measurement. This is reached in the most natural manner, as the preceding chapter closed with an account of Laguerre's projective definition of the angle. The hyperbolic case receives the fullest treatment, the fundamental

metrical formulæ are fully developed and discussed. I think that the student would be glad of a further discussion of non-euclidean trigonometry. The euclidean distance formula is shown as a *limiting* case of the non-euclidean; it is not a *special* case, as the author states on p. 118. There is also a small slip on p. 116 where we are told that in the hyperbolic plane "the locus of points equally distant from a straight line is not a straight line, but a pseudo circle with imaginary center and radius." It is called a "hypocycle." The radius is indeed imaginary, not so the center, which is real, but ultra-infinite, outside of the absolute conic. As for the term "hypocycle" I like it not. It is not generally used, and suggests an unholy compromise between the hypocycloid and the hypercycle of Laguerre.

Even more attractive material is given in Chapter VIII which deals with contact transformations in the plane, including the standard quadratic Cremona transformation, point-curve transformations, the pedal transformation, and the differential conditions for a contact transformation in terms of the variables x, y, p. I think it a great pity that some of these were not treated in greater detail. The Cremona transformation, which the author calls "quadric inversion," takes on a particular interest when explained as the interchange of isogonally conjugate points, the pedal transformation is beautifully factored into the product of a polar reciprocation and inversion in the same circle. Incidentally, I think that it would have been a good plan to give some discussion of polarizing in a circle. The applications are easy and attractive. The general form for a point transformation

$$\rho x_i' = f_i(x_1, x_2, x_3)$$

is meaningless unless the three homogeneous polynomials are of the same order. The statement (p. 140) that the polar of

$$\sum a_{ik}x_ix_k$$

is

$$\sum a_{ik}x_iy_k$$

is true only when

$$a_{ik} = a_{ki}.$$

Chapter IX, another interesting chapter, deals with the circle in tetracyclic coördinates. The author fails to point out (p. 150) that the four circles which are taken as the basis of a system of a general tetracyclic system must not be orthogonal to a fifth; i.e., must not be linearly dependent. I am not sure that a rearrangement of order, so as to begin with the idea of the power of a point with regard to a circle, and mutually orthogonal circles would not have led to a simpler treatment. The author is careful to show that with these coördinates we get a perfect continuum by adjoining not a line at infinity, but a different locus. This is a really most important idea for the young student to grasp, and it is well that Professor Woods treats it at such length. I confess, however, that I can not quite understand his statement (p. 142) that now "the infinite domain is a special straight line." In the real domain it is a single point, in the

complex, a pair of lines. The subject matter of this chapter is closely allied with that of Chapter XII which deals with the circle directly through its homogeneous coördinates, first the signless circle, then the oriented circle. The amount of fresh and attractive material presented to the student in these two chapters is surprising. The intervening Chapter XI is very short, and deals essentially with geometry on a real quadric surface.

The chapter on circle geometry is included among those dealing with three-dimensional geometries, for the totality of circles is a three-dimensional manifold. The more usual problems of three space are first taken up in Chapter XII. This deals with point and plane coördinates and is followed by chapters dealing with quadric surfaces, quaternary collineations, the sphere in cartesian and in pentaspherical coördinates. The spirit, method, and center of interest are exactly the same as in the plane, much of the development being a straight following of copy. As a pedagogical question, I can not be sure whether this involves a certain amount of lost motion, or whether such repetition is valuable to the student as a review, solidifying what he has already learned in the plane. For instance, the treatment of the angle of two lines in space is immediately based on the elliptic measurement of distances "in plano." The chapter on transformation in space includes three-dimensional non-euclidean measurement, Clifford parallels, and the differential equations for contact transformations. The Null System is mentioned on p. 248 but is not sufficiently explained either here or later, in connection with the linear line complex. I have an idea that a very small amount of kinematics would be welcome. For instance, after the equations for a rigid motion have been set up (p. 246) it is a very short task to show that when a rigid body is in motion, the relation at any instant between a point of the body and the plane through it perpendicular to the velocity vector is, in general, that of a null system. The student began his geometry with metrical theorems and, in my experience, he is always glad to go back to them now and then.

As the study of circle systems was used to lead up to three-dimensional space, so we move into four-space by the study of familiar figures, not spheres, as we should expect by analogy, but lines. We have a fairly full discussion of the Plücker line geometry, including linear and quadratic systems of lines. The treatment shows clearly the influence of Koenigs and Jessop, which is tantamount to saying that it is excellent. I find, on p. 308, the statement:

"*A line complex* . . . may be, but is not necessarily, defined by a single equation which is satisfied by the coördinates of the lines of the complex."

This statement is exactly true, but as the writer goes on in the next sentence to speak of the *order* of a line complex, it seems likely that he has an algebraic complex in mind. But in that case we have Klein's theorem[1] that every such complex may be expressed by a single equation in the coördinates of a line. It would have been better if some discussion had been given of the metrical and projective invariants of a linear line complex, in connection with the problem of finding the central axis and pitch, as is done, for instance, in Zindler's *Liniengeometrie*.[2]

[1] *Math. Annalen*, vol. 22.

[2] Vol. 1, Leipzig, 1902, p. 145.

In Chapter XVIII we have the coördinates of spheres, both signless and oriented. In connection with the latter we find an acute case of the trouble which arises when an author is not clear as to whether he is operating in the real or the complex domain. If the domain be complex, the distinction (p. 344) that a sphere with positive radius bounds its inside while one with negative radius bounds its outside is meaningless, but if the domain be real, the statement on p. 352 that there are ∞^2 spheres through a minimum line is erroneous. Rather more attention is given to systems of oriented spheres than would be warrantable were it not for the purpose of giving substance to the Line-Sphere transformation with which the chapter closes.

The last chapters deal with projective and euclidean geometry in a space of any number of dimensions. The treatment of the hyperquadric seems to me particularly good, although I note the absence of Sylvester's Law of Inertia which gives valuable information about real hyperquadrics.

It is a mistake for one viewing a landscape not to see the wood for the trees. It is a mistake to fix the attention on the minutiæ of a book, and lose sight of the general sweep. I have meticulously pointed out many small errors; it is the duty of an honest reviewer to do so. But now that I have them out of the way, I am free to say how much I admire the book as a whole. Professor Woods has undertaken the important and difficult task of introducing the student to that rich field of science called Modern Geometry. In my opinion, he has succeeded unquestionably better than any other writer using the English tongue. A book with which one might compare the present is Veblen and Young's *Projective Geometry*[1] which seems to include a good part of geometry, arithmetic, and abstract logic within its compass. But these authors consistently place logical considerations before didactic ones, and he who would approach higher geometry *for the first time* through their two closely written volumes must have much patience and courage, for "Jordan am a hard road to trabbel."

Years ago an enthusiastic French student said to me, in praise of Darboux's lectures:

"Il dit chaque jour quelque chose de nouveau, quelque chose d'intéressant."

This is essentially true of Professor Woods' book, and the student who passes rapidly from one new and interesting topic to another may well say, in current phrase: "Day by day, in every way, it's getting better and better."

J. L. COOLIDGE.

The Mathematical Theory of Probabilities. Volume I, Second edition. By ARNE FISHER. New York, The Macmillan Co., 1922. 8vo. 29 + 289 pages. Price $5.00.

It is especially gratifying that a man with a wide experience in the applications of probability to statistics, life insurance and the telephone business should write in the English language a treatise on theoretical probability and its applications. In an introductory note to the first edition, F. W. Frankland, one of

[1] Ginn and Co., Boston, vol. 1, 1910, vol. 2, 1918.

the foremost actuaries in America, wrote as follows: "While in French, in Italian, in German, in Danish, and in Dutch, scientific works on statistics were available galore, the dearth of such literature in the English language was little short of a national or racial scandal. With such works as those of Yule and Bowley, in recent years, there has been some possibility for the English-speaking student to acquire part of the knowledge needed. But it is hardly necessary to point out what a very large amount of new ground is covered by Mr. Fisher's new book as compared with such works as I have referred to."

These remarks apply, *a fortiori*, to the enlarged second edition. Mention should be made, however, of the recent appearance of J. M. Keynes's *A Treatise on Probability*, which will appeal strongly to those whose interest in probability is primarily philosophical or logical, this treatise being a continuation of the work of Boole and Venn. To those who want a text in English comparable to some extent in scope and general treatment with the standard books in German and French, Fisher's book is a godsend. The need of such a book is felt most keenly in those institutions where the study of Spanish has assumed such exaggerated proportions that a teacher of seniors or first-year graduates can not take for granted that his students can read either French or German. The reviewer has used the first edition of Fisher's book as a text, and found it highly satisfactory. The enlarged second edition should be doubly so. Problems for students, however, must be supplied by the instructor. For statistical experts, too, in certain forms of business, the book will be invaluable, as clearly indicated in an introductory note by Mr. M. C. Rorty, of the American Telephone and Telegraph Co. Mr. Rorty, after mentioning Mr. Fisher's skill in dealing with statistical curves, writes: ". . . Practical experience with these curves soon showed that in spite of minor errors, they were close enough to the real facts to make them of primary importance in traffic studies of all kinds and particularly in the development of mechanical switching devices. Their use for such purposes has now become a commonplace in telephone engineering."

Fisher's book is written in an entertaining style with numerous references, historical and critical. Special mention is made of Czuber's *Wahrscheinlichkeitsrechnung* as being lucid, terse, systematic, and attractive; and Fisher's book in the introductory portions bears some resemblance to that of Czuber. For example, Czuber's definition of mathematical probability is adopted—with quotation marks and reference—by Fisher.

The first five chapters of Fisher's book are devoted to definitions, explanations, and well-selected algebraic probability problems, ending with the St. Petersburg paradox. In Chapter VI inverse probability is discussed at length; absurdities arising from the indiscriminate use of Bayes's Theorem are pointed out, but this theorem is deemed valid when properly used. The exposition of Bayes's Theorem would have been more lucid if it had been divorced at the start from considerations of repeated trials. The law of large numbers is discussed in Chapter VII, and the relation between mathematical probabilities and relative frequencies. In Chapter VIII Stirling's Formula is deduced by Cesàro's elegant method,

which gives control of the error. As preparatory to this, the Wallis expression for π as an infinite product is obtained by the use of J_n, the integral from 0 to $\pi/2$ of $\sin^n x$. The argument on page 92 is, however, inconclusive; because J_n is an infinitesimal. But the desired result, Lim $J_{2m-1} \div J_{2m} = 1$, can be easily obtained from $J_{2m} < J_{2m-1} < J_{2m-2}$ and Lim $J_{2m-2} \div J_{2m} = 1$. In Chapter IX appear the theorems of Bernoulli, Poisson, and Tchebycheff. The next three chapters show how the actual dispersion may be compared with the theoretical dispersion to determine whether the series belongs to the Bernoullian type, to the Poisson type, or to the Lexis type, with normal, subnormal, or hypernormal dispersion, respectively. This is one of the most characteristic sections of Part I.

The next two parts, consisting of 120 pages, are an addition to the first edition. In the introductory historical survey, the theory of Laplace is deemed the proper starting point for statistical theory. Laplace, it is noted, was not a computer; and the less brilliant work of Gauss received the greater attention. The theory of Laplace, however, has been rejuvenated, and brought to greater use by such writers as Charlier, who supplied suitable forms for computation. Fisher mentions the valuable work of Karl Pearson; but regards his method, which involves closed expressions, as less flexible and effective than the developments in series preferred by Scandinavian and German writers. In another publication, Fisher hopes to give some still more advanced probability theories. Here, no doubt, reference will be made to the monumental work of R. von Mises,[1] who, by axioms on sequences, eliminates the mystery of "equally probable cases," and by an elaboration of the Czuber method eliminates the troublesome *a priori* probability used in Bayes's Theorem.

Part II gives the mathematical theory of statistical series. By the use of the Fourier Integral Equation, a very general form of frequency function is set up as an integral involving the semi-invariants of Thiele. If only the first two of these semi-invariants are used, the frequency function is represented by the well-known "normal" probability curve. A general frequency function is then expressed as a series in terms of the Hermite polynomials and the factor $e^{-x^2/2}$, the coefficients c_i being integrals involving Hermite polynomials. The coefficients are also expressed in terms of the semi-invariants, and by proper choice of origin and unit, $c_1 = 0 = c_2$.

Part III is devoted primarily to applications of the theory, with numerical computations in detail. Here appear, also, brief expositions of the method of least squares, of logarithmically transformed frequency functions, of the Charlier B Curve, and of the law of small numbers. Of special interest is the graduation of the American Men Mortality Table ($AM^{(5)}$) by means of a compound Charlier curve.

There is a table of the Laplacean probability function, certain of its derivatives, and its integral. But no index of titles or authors is given. This would have been especially useful, because of the very numerous references appearing throughout the book.

[1] (1) *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, (2) *Grundlagen der Wahrscheinlichkeitsrechnung*, *Mathematische Zeitschrift* (1919), vol. 4, pp. 1–97; vol. 5, pp. 52–99.

Though statisticians are by no means agreed as to where emphasis should be placed, it may safely be said that Fisher's book covers a wide range of very important topics. And, in the reviewer's opinion, it is the only book in English comparable with the standard works on probability in German, French, and Italian.

E. L. DODD.

Real Mathematics. By ERNEST G. BECK. Froude, Hodder and Stoughton, London, 1922. 306 pages.

The motive of the author is described in the preface as follows:

"One of the principal objects of this book is to offer assistance to the practical engineer and to engineering students in the acquisition of a real, serviceable and sound mathematical equipment. The book is intended to augment the standard textbooks and orthodox methods of study. The desire is rather to bring about a change of attitude towards mathematics than to propose methods which shall merely be different from those in use—to show the thing as an actual, tangible reality, instead of as a collection of rigid and unrelated rules and formulæ: and to persuade students to touch and handle it for themselves with confidence and understanding instead of regarding it from afar as some rather awful and totally incomprehensible abstraction."

In his endeavor to harmonize the modern demands with the best traditions of the various topics, the author has been very successful. He has treated carefully the foundation of each topic, and gives numerous examples to explain further the difficulties involved. In all cases, in order that the reader may learn principles and gain power, fundamental methods are set forth and applied to problems.

The author has shown good pedagogical insight by trying to make the reader feel at the beginning of each topic under consideration that a worth-while problem requiring solution is before him. Interest is aroused immediately, and the reader is more than anxious to follow up the carefully planned explanation.

The book is well illustrated, is written in an interesting style and will be valuable not only to the engineer, but also to any teacher of mathematics.

The subject matter takes up the usual topics given during the first two years of college mathematics. About half of the book is devoted to Arithmetic, Geometry, and Trigonometry, and the remainder to the Calculus.

To sum up, it seems to the reviewer that the author has been very successful in his effort to enliven and enrich the subject and to present it in such a manner as to develop in the reader an exact mathematical imagination.

C. N. MILLS.

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AMERICAN JOURNAL OF MATHEMATICS, volume 44, no. 3, July, 1922 (published February, 1923): "Plane involutions of order four" by T. R. Hollcroft, 163–171; "Differential equations with a continuous infinitude of variables" by I. A. Barnett, 172–190; "The factorization of the rational primes in a cubic domain" by G. E. Wahlin, 191–203; "On the structure of finite continuous groups with a single exceptional infinitesimal transformation" by S. D. Zeldin, 204–216; "The Laplace-Poisson mixed equation" by K. P. Williams, 217–224; "The four color problem" by P. Franklin, 225–236.

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Richardson, 377–387; "A property of certain functions whose Sturmian developments do not terminate" by O. D. Kellogg, 388–389; "Relating to the proof of an existence theorem for a certain type of boundary value problem" by H. I. Davis, 390–394; "Note concerning the roots of an equation" by K. P. Williams, 394–396; "The complete existential theory of Hurwitz's postulates for Abelian groups and fields" by B. A. Bernstein, 397–399; "On Kummer's memoir of 1857 concerning Fermat's last theorem" by H. S. Vandiver, 400–407; Review by D. Jackson of H. Hahn, *Theorie der reellen Funktionen*, Vol. 1 (Berlin, 1921), 408–411, "Shorter notices," 412–417 [Reviews: by E. B. Wilson of J. H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge, 1920) and *Problems of Cosmogony and Stellar Dynamics* (Cambridge, 1919); by E. B. Cowley of G. Loria, *Storia della Geometria Descrittiva dalle Origini sino ai Giorni nostri* (Milano, 1921); by F. M. Morgan of H. Hilton, *Plane Algebraic Curves* (Oxford, 1920); by J. B. Shaw of D. Bhattacharyya, *Vector Calculus* (Calcutta, 1920); by J. W. Young of L. Bolton, *An Introduction to the Theory of Relativity* (New York, 1921); by C. Sisam of L. Berzolari, *Geometria analitica*, Part 1, *Il Metodo delle Coordinate* (2d ed. Milano); by F. Cajori of the Gauthier-Villars reprints of memoirs by Laplace on *Probabilities*, by Lavoisier and Laplace on *Heat*, and by Ampère on *Electromagnetism and Electrodynamics;* and by L. C. Mathewson of L. Baumgartner, *Gruppentheorie* (Berlin, 1921)]; Notes, 418–420; New publications, 421–424—No. 9, December, 1922: "Condition that a tensor be the curl of a vector" by L. P. Eisenhart, 425–427; "A new generalization of Tchebycheff's statistical inequality" by B. H. Camp, 427–432; "Two theorems on multiple integrals" by P. Franklin, 433–435; "Kirkman parades" by F. N. Cole, 435–437; "Impossibility of restoring unique factorization in a hypercomplex arithmetic" by L. E. Dickson, 438–442; "A revision of the Bernoullian and Eulerian functions" by E. T. Bell, 443–450; "On the existence of curves with assigned singularities" by J. L. Coolidge, 451–455; "A generalization of normal congruences of circles" by J. L. Walsh, 456–462; "A correction" by E. Hille, 462; Reviews: by C. N. Moore of A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre* (Vol. 1, 3d part, Leipzig and Berlin, 1921), 463–465, and of A. Wangerin, *Theorie des Potentials und der Kugelfunktionen* (Vol. 2, Berlin and Leipzig, 1921), 465; by G. E. Wahlin of L. Bianchi, *Lezioni sulla Teoria dei numeri algebrici e Principi d'Aritmetica analitica* (Pisa, 1921), 466, and of E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig and Berlin, 1918), 466–467; by H. J. Ettlinger of L. Bieberbach, *Lehrbuch der Funktionentheorie* (Vol. 1, Leipzig, 1921) and *Funktionentheorie* (Leipzig, 1922), 467–468, and of H. Burkhardt, *Funktionentheoretische Vorlesungen* (Berlin: Vol. 1, part 1, *Algebraische Analysis*, 3d edition, 1920; part 2, *Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen*, 5th edition, 1921), 475; by O. D. Kellogg of H. Rothe, *Vorlesungen über höhere Mathematik* (Vienna, 1921), 468–469; by J. W. Young of L. B. Benny, *Plane Geometry* (Glasgow and Bombay, 1922), 469, and of F. F. P. Bisacre, *Applied Calculus* (Glasgow and Bombay, 1921), 471; by W. C. Graustein of G. Bouligand, *Cours de géométrie analytique* (Paris, 1919), 470–471; by C. L. E. Moore of A. Lotze, *Die Grundgleichungen der Mechanik insbesondere der starren Körper* (Leipzig, 1922), 472; by L. C. Mathewson of J. Poirée, *Précis d'arithmétique* (Paris, 1921), 472–473; by L. W. Dowling of O. S. Adams, *Latitude Developments connected with Geodesy and Cartography* (Washington, 1921), and of C. H. Deetz and O. S. Adams, *Elements of Map Projection* (Washington, 1921), 473; by J. N. Van der Vries of F. Schuh, *Lessen over de Hoogere Algebra* (Vol. 1, Groningen, 1920), 474; by A. D. Pitcher of *Encyklopädie der Mathematischen Wissenschaften*, Vol. 2, part 2 (Leipzig, 1901–1921), 474; and by A. Emch of M. Emanaud, *Géométrie perspective* (Paris, 1921), 475; Notes, 476–478; New publications, 479–483; Thirty-first annual list of papers read before the American Mathematical Society and subsequently published, 484–492. [The number of papers listed is 153, by 97 authors, the highest number of papers by one author being 11; 16 authors have contributed 3 or more papers each to a total of 63 papers]; Index of Volume 28, 493–503.

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Journal of the Washington Academy of Sciences, volume 11, no. 19, November 19, 1921: "On the correlation between any two functions and its application to the general case of spurious correlation" by L. J. Reed, 449–455.—Volume 12, no. 1, January 4, 1922: "A mathematical note on the annealing of glass" by E. D. Williamson, 1–6.

L'Enseignement Mathématique, volume 22, no. 5 (published January, 1923): "Un chapitre de méthodologie mathématique, les imaginaires de Galois" by M. Stuyvaert, 249–268;

"Sur les radicaux carrés" by B. Niewenglowski, 269–277; "Démonstration du théorème de Liouville par l'élimination du temps entre les équations de Lagrange" by E. Turrière, 277–285; "La section mathématique de l'institut normal supérieur de Bolivie" by C. Lurquin, 286–290; "Chronique," 290–304 [contains among other things a summary of colloquium lectures given before the Société Mathématique Suisse at Bienne, April 23, 1922, by E. Hecke of Hamburg, on Arithmetic and the Theory of Functions; by M. Plancherel of Zürich, on Difference Equations and Differential Equations; and by W. Blaschke of Hamburg, on Selected Chapters in Differential Geometry]; "Notes et Documents," 304–311; "Bibliographie," 311–324 [Reviews: by A. Buhl of P. Appell, *Education et Enseignement* (Paris, 1922), of E. Goursat, *Leçons sur le Problème de Pfaff* (Paris, 1922), and of B. Levi, *Abbaco da 1 a 20* (Parma, 1922), "Une œuvre italienne digne d'être imitée en toutes les langues"; by R. Wavre of H. Bergson, *Durée et simultanéité* (Paris, 1922), of G. Juvet, *Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu* (Paris, 1922), of P. Lévy, *Leçons d'Analyse Fonctionelle* (Paris, 1922), and of Ch.-J. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale* (4th ed., 2 vols., Louvain and Paris, 1921–1922); and by D. Mirimanoff of L. Bieberbach, *Lehrbuch der Funktionentheorie* (Vol. 1, Leipzig and Berlin, 1921)]; "Bulletin Bibliographique," 325–336 [short notices of 28 mathematical books, and lists of articles in current periodicals and of doctors' theses].

Mathematische Annalen, volume 88, nos. 1 and 2, December, 1922: "Über die Komposition der quadratischen Formen" by A. Hurwitz, 1–25; "Über die Anzahl der Klassen positiver ternärer quadratischer Formen von gegebener Determinante" by A. Hurwitz, 26–52; "Zur Theorie der Polynomideale und Resultanten" by K. Henzelt and E. Noether, 53–79; "Algebraische Theorie der Ringe" by W. Krull, 80–122; "Zur Theorie der Kleinschen Ergänzungsrelationen" by H. Falckenberg, 123–135; "Zur Parametrixmethode" by O. Haupt, 136–150; "Die logischen Grundlagen der Mathematik" by D. Hilbert, 151–165 [Quotation: "Meine Untersuchungen zur Neubegründung der Mathematik bezwecken nichts Geringeres, als die allgemeinen Zweifel an der Sicherheit des mathematischen Schliessens definitiv aus der Welt zu schaffen"]; "Eine Bemerkung zu der Arbeit des Herrn Bieberbach 'Über die Verteilung der Null- und Einstellen analytscher Funktionen'" by M. Fekete, 166–168.

School Review, volume 30, no. 7, September, 1922: "Using home-made tests in high schools" by L. Byrne, 536–546 [describes certain tests in algebra and geometry, and draws conclusions in harmony with the recommendations of the National Committee on Mathematical Requirements]; "Higher mathematics in high school" [note on N. B. Rosenberger, *The Place of Elementary Calculus in the Senior High School Mathematics* (New York, 1921)] by E. R. Breslich, 552–553.—No. 10, December: Note by E. R. Breslich on G. Wentworth, D. E. Smith and H. D. Harper, *Fundamentals of Practical Mathematics* (Boston, 1922), 792.—Volume 31, no. 1, January, 1923: Note by E. R. Breslich on the Cleveland Board of Education's *Course of Study and Syllabus: Junior High School Mathematics* (Cleveland, 1922), 70–71; Note by C. A. Stone on W. D. Reeve, *General Mathematics, Book Two* (Boston, 1922), 75–76.

Scientific Monthly, volume 15, no. 6, December, 1922: "Easy group theory" by G. A. Miller, 512–519.—Volume 16, no. 1, January, 1923: "The theory of relativity and its influence on scientific thought" by A. S. Eddington, 34–53.—No. 2, February: "The mathematical sciences in the Latin colonies of America" by F. Cajori, 194–204.

Transactions of the Cambridge Philosophical Society, volume 22, no. 18, February, 1920: "The hydrodynamical theory of the lubrication of a cylindrical bearing under variable load, and of a pivot bearing" by W. J. Harrison, 373–388.—No. 19, May, 1920: "On the integers which satisfy the equation $t^3 \pm x^3 \pm y^3 \pm z^3 = 0$" by H. W. Richmond, 389–403.—Nos. 20 and 21, August, 1920: "On cyclical octosection" by W. Burnside, 405–411; "Congruences with respect to composite moduli" by P. A. MacMahon, 414–424.—No. 22, October, 1920: "On the stability of the steady motion of viscous liquid contained between two rotating coaxal circular cylinders" by K. Tamaki and W. J. Harrison, 425–437.—No. 23, May, 1922: "On a general infinitesimal geometry, in reference to the theory of relativity" by W. Wirtinger, 439–448.—No. 24, May, 1922: "On the fifth book of Euclid's Elements" by J. M. Hill, 449–462; [Quotation: "The object of this paper is to endeavor to recover the train of thought which led the writer (supposed to be Eudoxus, but whom I will refer to as Euclid), of the above-mentioned work, to the formulation of his Fifth Definition (the test for the sameness of two ratios), and his Seventh Definition (the test for distinguishing the greater from the smaller of two unequal ratios)."—No. 25, June, 1922: "The influence of electrically conducting material within the earth on various phenomena of terrestrial magnetism" by S. Chapman and T. T. Whitehead, 463–482.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF ALBION COLLEGE, Albion, Mich.
[*1921*, 322.]

As officers for the first semester of the year 1922–1923, the following were elected: President, Judson Foust '23; vice-president, Norma Sleight '24; secretary-treasurer, Sybil Davidson '24; member of program committee, Gordon Wheeler '24: as officers for the second semester: President, Gordon Wheeler '24; vice-president, Cleon Richtmeyer '24; secretary-treasurer, Mary Winegar '23; member of program committee, Walter Moore '24.

The following papers were read:

October 17, 1922: "A bibliography of uses and value of mathematics" and "Contribution of mathematics to the world's progress" by Sybil Davidson '24; "Mathematics applied to professions" by Cleon Richtmeyer '24; "The cultural value of mathematics" by Mary Winegar '23.

November 7: "Human worth of rigorous thinking" by Lester Chamberlain '24; "Mathematics in the United States previous to 1800" by Gordon Wheeler '24.

November 20: "Thirteen proofs of the Pythagorean Theorem" by Evelyn Scott '24 and Elvah Dayton '24.

December 24: "Psychology of errors in mathematics" by Evelyn Palmatier '24; "Psychology of the equation" and "Psychology of problem solving" by Norma Sleight '24.

January 8, 1923: "The roots of the quartic equation" by Judson Foust '23.

January 23: "Ruler-compass construction of regular polygons" by Harold Black '23.

February 26: "Development and application of hyperbolic functions" by Walter Moore '24; "A discussion of circular functions" by William Plumb '24.

March 2: Roll call—Who's Who in mathematics. "Selling mathematics" (presidential address) by Gordon Wheeler '24; "Some of the difficulties of calculus teaching" by Professor E. R. Sleight.

March 16: "Confocal conics" by Mary Bachelor '24; "Some common fallacies in mathematics" by Frances Howlett '23.

April 17: Roll call—Theorem with name of discoverer. "Hyperspace" by Helen Camburn '24; Discussion of fourth dimension—Professor E. R. Sleight, leader.

May 1: "The nth root of arithmetical numbers" by Mary Winegar '23; "Magic squares" by Frances Howlett '23.

May 15: A mathematician noted in some other line. "History and transcendency of π" by Cleon Richtmeyer '24; "Astrology" by Sybil Davidson '24.

May 29: Social evening and election of officers.

(Reported by Professor E. R. Sleight.)

THE MATHEMATICAL CLUB, HARVARD UNIVERSITY, Cambridge, Mass.
[*1921*, 274.]

The following officers were elected for 1922–1923: President, H. W. Brinkman, Gr.; secretary-treasurer, B. Z. Linfield, Gr.; faculty adviser, Dr. Joseph L. Walsh. Papers were read as follows:

October 11, 1922: "Elementary definitions of the trigonometric functions" by Professor W. F. Osgood.

October 24: "Continuation of analytic functions" by M. H. Stone, instructor.

November 7: "The four color map problem" by Dr. Philip Franklin, instructor.

November 21: "Asymptotic series" by Bernard Koopman, instructor.

December 5: "A mathematician's tour in Europe" by Professor R. C. Archibald, of Brown University.

December 19: "Automorphic functions" by Isador Sheffer '23.

January 5, 1923: "Complex plane analytics in 4-space" by Dr. James Taylor, of the Massachusetts Institute of Technology.

February 20: "Nets and the Dirichlet problem" by Dr. Norbert Wiener, of the Massachusetts Institute of Technology.
March 6: "The equations of elasticity" by Julian Holley, Gr.
March 27: "A method of series in elasticity" by Carl Garabedian, Gr.
April 10: "Some problems in algebraic geometry" by Professor J. L. Coolidge.
May 1: "Squaring the circle" by David Widder, instructor.
May 15: "The mathematical works of Professor Charles Leonard Bouton" by Professor W. C. Graustein.

(Reported by B. Z. Linfield.)

The Mathematics Club of Hood College, Frederick, Md.

On December 7, 1922, the Mathematics Club of Hood College was organized to stimulate interest in Mathematics and to present to the students extra-curriculum topics. All students taking elective courses in mathematics are eligible to membership. Goldae Biser '23 was elected president; Miss Margaret Packer, instructor, is acting secretary; Professor Lillian O. Brown is faculty adviser. The following is the program for the year 1922–1923:
December 7, 1922: "Anecdotes about famous mathematicians" by all members.
January 12, 1923: "Mathematics and life activities" by Professor Clara Bacon of Goucher College.
February 23: "Three classical geometric problems" by Dorothy Eyler '23, Pauline Beachley '23, and Sarah Markley '23.
March 16: "Hindu arabic numerals" by Goldae Biser '23; "Development of mathematical symbols" by Hazel Zimmerman '23.
April 20: "Mathematical prodigies" by Ruth Feaga '24 and Ruth Michael '24; "Curiosities of numbers" by Helen Goodfellow '24 and Grace Allen '24.
May 18: "Women in mathematics" by the sophomore members.
June 2: Picnic.

(Reported by Miss Packer.)

The Junior Mathematics Club of the University of Wisconsin, Madison, Wis.

[*1921*, 391.]

The Junior Mathematics Club, with a membership of thirty, held meetings in 1922–1923 as follows:
October 26, 1922: General discussion and business meeting.
November 9: "Music and mathematics" by Professor A. Dresden.
November 23: "Mathematical logic" by Frank Bruner '25; "Relation of high school mathematics to college algebra" by Reinhard Hein '26.
December 7: "Theory of probability" by Professor L. W. Dowling.
March 8, 1923: "Geometry in the compass plane" by Pearl Anderberg '23.
March 22: "Hyperbolic synthetic geometry" by Elizabeth Baird '23.

(Reported by Miss Viola Jenson, Secretary.)

TO THE COMPLEX VARIABLE.

By Josephine Reddish, Knoxville, Tenn.

As I pore o'er your pages by lamplight,
Oh, wonderful book of my dreams,
I can scarce see the ρ's and the θ's
For the glint of those dazzling beams.

The x's and y's are commingled,
Confused with the u's and the v's;
And my mind and my heart are a-tremble
With the pangs of some strange new dz's.

For there on the pages before me,
'Mid the lore of the ancient and wise,
Gleams with wondrous unquenchable brilliance
The light of those marvelous i's.

When weary from long hours of labor,
I find myself starting to nap,
And my mind refuses to function
In the maze of a conformal map,

'Round the curves of the equipotential,
Coursing in on the long lines of flow,
From regions divine, holomorphic,
Comes stealing a quickening glow.

I cast off my growing inertia,
Fickle goddess in complex disguise,
As I bask with the heart of a lover,
In the light of your wonderful i's.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3025. Proposed by E. H. CLARKE, Hiram College.

Sum the infinite series

$$1 + \frac{2^k x^2}{2!} + \frac{3^k x^4}{4!} + \frac{4^k x^6}{6!} + \cdots \frac{n^k x^{2n-2}}{(2n-2)!} + \cdots,$$

k a positive integer.

3026. Proposed by B. F. FINKEL, Drury College.

A flat board 12 ins. square is suspended in a horizontal position by strings attached to its four corners, A, B, C, D, and a weight equal to the weight of the board is laid upon it at a point 3 ins. distant from the side AB and 4 ins. from AD; find the relative tensions in the four strings.

From Bowser's *Analytical Mechanics*, p. 92.

Is this a determinate problem? If not, why not?

3027. Proposed by C. N. SCHMALL, New York City.

A parabola whose base (double ordinate) is h and altitude k has a circle inscribed of diameter d, and a circle circumscribed, of diameter D. Show that $D + d = h + k$. Note that k can not be $< h/2$.

3028. Proposed by NORMAN ANNING, University of Michigan.

Equilateral triangles are described on the sides of a right triangle. Dissect the triangles on the legs and reassemble the parts to form the triangle on the hypotenuse.

3029. Proposed by J. ROSENBAUM, Milford, Conn.

To locate two points, D and E, on the sides AB and BC of a triangle ABC such that $AD : DE : EC$ shall be equal to $p : q : r$, where p, q, and r are given line segments.

The above is a generalization of problem 2816 (*1920*, 134).

3030. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the envelope of the bisector of the angle that a given segment subtends at a variable point of a given line.

3031. Proposed by S. A. COREY, Des Moines, Iowa.

On page 183 of the 1922 MONTHLY, Professor Bennett has given an expression for the limit of error in evaluating a definite integral by Simpson's rule. Give a similar expression for limit of error when evaluation is made by Weddle's rule; also when evaluation is made by the similar but somewhat more accurate formula:

$$\int_0^x f(x)dx = \frac{x}{840}[41(y_0 + y_6) + 216(y_1 + y_5) + 27(y_2 + y_4) + 272y_3].$$

(See MONTHLY, June–July, *1912*, 125, and Feb., *1918*, 88.)

3032. Proposed by OTTO DUNKEL, Washington University.

If $a_1, a_2, \cdots, a_n$ are any real or complex quantities which satisfy the equation

$$x^n - na_1x^{n-1} + {}_nC_2a_2^2x^{n-2} + \cdots + (-1)^i{}_nC_ia_i^ix^{n-i} + \cdots + (-1)^na_n^n = 0,$$

where ${}_nC_i = n!/(n-i)!i!$, prove that $a_1 = a_2 = \cdots = a_n$.

3033. Proposed by A. A. BENNETT, University of Texas.

Prove the existence in the inversion plane, for an arbitrary positive integer n, of a configuration of 2^{n-1} points and 2^{n-1} circles, n of the points lying on each circle and n of the circles passing through each point. Show that an arbitrary set of n points entirely unrestricted save for distinctness may be taken as points of the configuration, and that there will then remain two degrees of freedom in the construction of the figure. State the Euclidean theorem in case one of the points is at infinity. Show also that in case n concurrent circles of the configuration are congruent, all the circles are congruent.

3034. Proposed by J. L. RILEY, Stephenville, Texas.

If every root of the equation $f'(x) = 0$ be subtracted from every root of the equation $f(x) = 0$, find the sum of the reciprocals of the differences.

SOLUTIONS.

2871. [1921, 36]. Proposed by the late L. G. WELD.

Weight being disregarded, a package may be admitted to the parcels post if the length plus the greatest girth, measured transversely to the length, does not exceed 72 inches. What is the size of the smallest square window through which all admissible rectangular boxes can be passed?

SOLUTION BY A. A. BENNETT, University of Texas.

Let a, b, c be the dimensions of the rectangular box where $a \geqq b \geqq c$. Then a is the length, and $2(b + c)$ the girth. Thus $a + 2(b + c) \leqq 72$ is the condition imposed by law for rectangular boxes. A box may be fitted snugly through a square window in each of two ways. The length does not enter this question. In one way the sides of the box are parallel to those of the window and the smallest square window has each edge e equal to b. In the other way the box fits in diagonally and the square window is readily seen to have each edge e equal to $(b + c)/\sqrt{2}$. The most troublesome box is clearly that of the most troublesome cross section so that the length a, by definition not less than b, must be a minimum that is equal to b in the worst case. For boxes fitted in snugly obliquely, the edge of the square window is proportional to the girth so that $b + c$ must be a maximum or, a, a minimum. One arrives then at a cubical box, $a = b = c$. But this does not give the solution, since a cubical box by being put in parallel to the edges of the window will go through a smaller square window. The worst case is, therefore, one in which the box fits snugly in each way, diagonally and parallel, so that $e = b = (b + c)/\sqrt{2} = a$, or $a + 2(b + c) = (1 + 2\sqrt{2})e = 72$, and the square window has each edge equal to $72/(1 + 2\sqrt{2})$ inches or 18.807 − inches and an area of 353.703 sq. in.

The problem may be extended by asking for the *rectangular* window of smallest area. One's first thought might be that no answer could be obtained. However, the smaller edge of the window must be $14\frac{3}{4}$ inches since this is only just large enough to admit a cubical box where $a = b = c$ and $a + 2(b + c) = 72$. The other, longer edge satisfies an equation obtained by a discussion analogous to that above.

For convenience denote the shorter edge of the window by e, and suppose the longer edge to be equal to b. The box of cross section $b \times c$ when just fitted in diagonally will divide the side of length b into segments $2bc^2/(b^2 + c^2)$ and $b(b^2 - c^2)/(b^2 + c^2)$ and the side e in segments $2b^2c/(b^2 + c^2)$ and $c(b^2 - c^2)/(b^2 + c^2)$. Further $3b + 2c = 5e$ when $a = b$; also $2b^2c/(b^2 + c^2) + c(b^2 - c^2)/(b^2 + c^2) = e$. Thus $(c/e) = \frac{5}{2} - \frac{3}{2}(b/e)$. Substituting in $(c/e)[3(b/e)^2 - (c/e)^2]/[(b/e)^2 + (c/e)^2] = 1$, we have $9(b/e)^3 + 101(b/e)^2 - 285(b/e) + 175 = 0$, from which the trivial solution $b/e = 1$ may be at once removed, leaving $9(b/e)^2 + 110(b/e) - 175 = 0$. Whence $b = 16\sqrt{46} - 88$, or 20.517 + in. for the longer edge of the rectangular window, giving an area of only 295.445 − sq. in. for the smallest rectangular window.

NOTE BY OTTO DUNKEL, Washington University, and H. P. MANNING, Providence, R. I.—

We let $e = 72/5$ and $b_1 = (10\sqrt{46} - 55)e/9$. Then a box in which $b = b_1$ will fit diagonally in a window whose dimensions are $b_1 \times e$. Professor Bennett has proved this, but he has not proved that all boxes can be passed through such a window.

Let $u \times e$ be the dimensions of a window in which a box of cross section $b \times c$ (where $3b + 2c = 5e$) will just fit diagonally. u must satisfy the equation $(b - c)^2(u + e)^2 + (b + c)^2(u - e)^2 - 2(b^2 - c^2)^2 = 0$ (see *1920*, 327). Eliminating c we have

$$f(u, b) = 50(b - e)^2(u + e)^2 + 2(5e - b)^2(u - e)^2 - 25(b - e)^2(5e - b)^2 = 0. \qquad (1)$$

If $b = e$, $u = e$. If $b > e$ and $< (5 - 2\sqrt{2})e$ this equation has one root less than e and one root greater than e; for $f(e, b) = 25(b - e)^2[8e^2 - (5e - b)^2]$, which is negative. It is the larger value of u that belongs to our problem.

For $u = b$, f becomes $(b - e)^2[50(b + e)^2 - 23(5e - b)^2]$, which is zero when $b = b_1$. Then if $b < b_1$ this is negative and the value of u which is greater than e is greater than b, and if $b > b_1$ this is positive and the value of u which is greater than e is less than b. In the former case the box passes through a smaller window when it is put through in the first way, with its sides parallel to the sides of the window, than when passed through in the second way. In the second case the reverse is true.

It will now be shown that in the second case u is also less than b_1, and therefore that all boxes for which $b > b_1$ will pass diagonally through a window whose dimensions are $b_1 \times e$.

For $b = 5e/3$, f reduces to $\frac{4}{9}\frac{0}{0}e^2(u^2 - \frac{1}{9}\frac{6}{0}e^2)$. That is, $u = 4e/3$, which is less than b_1. Now the coefficient of u^2 in $f(u, b)$ does not vanish for any real value of b, and the two roots of the equation $f(u, b)$ are continuous single-valued functions of b in the interval $e < b < 5e$. When b varies from $5e/3$ to b_1, the value of u which is greater than e will remain less than b_1, or for some particular value of b will become equal to b_1.

When u equals b_1, f becomes

$$\varphi(b) = 50(b - e)^2(b_1 + e)^2 + 2(5e - b)^2(b_1 - e)^2 - 25(b - e)^2(5e - b)^2.$$

One root of $\varphi(b)$ is b_1. We shall prove that $\varphi(b)$ does not become zero for any value of b between b_1 and $5e/3$.

$\varphi(0) = 25e^2(4b_1^2 - 21e^2)$, which is negative.
$\varphi(e) = 32e^2(b_1 - e)^2$, which is positive.
$\varphi(b_1) = 0$.
$\varphi'(b_1) = 4(b_1 - e)(b_1^2 + 244b_1e - 345e^2)$, which is positive.

This shows that when b passes from e to b_1, φ must first be negative in order to increase to zero.

$\varphi(5e/3) = \frac{4}{9}\frac{0}{0}e^2(b_1^2 - \frac{1}{9}\frac{6}{0}e^2)$, which is positive.
$\varphi(\infty)$ is negative.

φ then has one root between 0 and e, one between e and b_1, one at b_1, and one greater than $5e/3$. All of the roots of φ are accounted for, and this proves that it does not become zero between b_1 and $5e/3$.

In other words, for no value of b between b_1 and $5e/3$ can u become equal to b_1, and as u is less than b_1 when $b = 5e/3$, it must remain less than b_1 for all these values of b. All boxes can pass through the window of dimensions $b_1 \times e$, those for which $b \leqq b_1$ by the first way, and those for which $b \geqq b_1$ by the second way.

The shorter side of the window must be at least equal to e. When the shorter side is equal to e the longer side must be equal to b_1. Now if the shorter side is made a little longer, the longer side may be made a little shorter, at least for some boxes. It has not been shown that we cannot in this way make the area of the window smaller. It will be sufficient to show that a window in which a box will fit snugly both ways must have a larger area when its shorter side is of length $v > e$, for with a given v no smaller window will admit this particular box.

Writing $u + v$ and $u - v$ in (1) for $u + e$ and $u - e$, putting $u = b$, and, finally eliminating b by putting $z = bv$, we find that z will be a function of v whose equation may be written

$$\left(\frac{z + v^2}{5ev - z}\right)^2 + \frac{1}{25}\left(\frac{z - v^2}{z - ev}\right)^2 - \frac{1}{2} = 0.$$

Now if we let U and V denote the expressions in the two parentheses we can determine their signs and the signs of their derivatives with respect to z and v, and so prove that

$$\frac{dz}{dv} = -\frac{25U\frac{\partial U}{\partial v} + V\frac{\partial V}{\partial v}}{25U\frac{\partial U}{\partial z} + V\frac{\partial U}{\partial z}} > 0.$$

This will complete the proof that the window of dimensions $b_1 \times e$ is the rectangular window of smallest area that will admit all possible boxes.

Also solved by T. M. BLAKSLEE and F. L. WILMER.

2958 [1922, 82]. Proposed by R. P. BAKER, University of Iowa.

Over a frictionless pulley a weightless cord sustains at one end a mass M, while the other end is wound on the axle of a wheel of mass M and moment of inertia N. At the zero of time the wheel revolves with angular velocity ω and tends to wind up the cord. Describe the motion neglecting friction.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let the vertical distances from some fixed horizontal line in the plane of motion *down* to the mass M and to the wheel be x and x_1, respectively. Let θ be the angle turned through by the wheel in unwinding the cord and let T be the tension of the cord. Then we have the geometric condition that $x + x_1 - r\theta = \text{const.}$, r being the radius of the axle. For the equations of motion we have

$$Mg - T = M\ddot{x}, \qquad Mg - T = M\ddot{x}_1, \qquad Tr = N\ddot{\theta}.$$

These equations give

$$\ddot{\theta} = \frac{2Mgr}{2N + Mr^2}; \qquad \text{whence} \qquad \dot{\theta} = \frac{2Mgr}{2N + Mr^2}t - \omega,$$

showing that the cord will wind up for $\frac{(2N + Mr^2)}{2Mgr}\omega$ seconds and thereafter unwind. Again

$$\ddot{x} = g - \frac{T}{M} = \frac{gMr^2}{2N + Mr^2} \qquad \text{and} \qquad x = \frac{gMr^2}{2(2N + Mr^2)}t^2.$$

That is, the mass M will descend with constant acceleration. Further,

$$\ddot{x}_1 = \ddot{x} = \frac{gMr^2}{2N + Mr^2} \qquad \text{and hence} \qquad \dot{x}_1 = \frac{gMr^2}{2N + Mr^2}t - r\omega.$$

That is, the wheel will rise until $t = \frac{2N + Mr^2}{gMr}\omega$. Now if a is the initial value of x_1,

$$x_1 = \frac{gMr^2}{2(2N + Mr^2)}t^2 - r\omega t + a \qquad \text{and} \qquad a - x_1 = \frac{2N + Mr^2}{2Mg}\omega^2 \qquad \text{where} \qquad t = \frac{2N + Mr^2}{gMr}\omega.$$

That is, the wheel will rise a distance $\frac{2N + Mr^2}{gMr}\omega^2$ above its first position and will thereafter descend with constant acceleration.

Also solved by PHILIP FITCH and WILLIAM HOOVER.

2965 [1922, 130]. Proposed by C. N. MILLS, Tiffin, Ohio.

If a quadrilateral inscribed in a square has the diagonals a and b and the area A, show that the area of the square is $\frac{a^2b^2 - 4A^2}{a^2 + b^2 - 4A}$.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let φ be the angle between b and a side of the square into which it projects and let θ be the angle between a and b. We will then have, since either equals the side of the square,

$$a \sin(\theta + \varphi) = b \cos\varphi; \qquad \text{whence} \qquad \tan\varphi = (b - a\sin\theta)/a\cos\theta$$

and, if S is the area of the square,

$$S = b^2 \cos^2 \varphi = a^2b^2 \cos^2 \theta/(a^2 + b^2 - 2ab \sin \theta).$$

But $A = \frac{1}{2}ab \sin \theta$; whence $S = (a^2b^2 - 4A^2)/(a^2 + b^2 - 4A)$.

Also solved by J. Q. McNatt, A. Pelletier and A. V. Richardson.

2967 [1922, 179]. Proposed by ELIJAH SWIFT, University of Vermont.

A plane revolves about one of two non-coplanar lines as an axis. Find the locus, *in the plane*, of the intersection of the plane and the other line.

Solution by R. S. Underwood, Alabama Polytechnic Institute.

Let the x-axis be the axis of rotation and the z-axis, the common perpendicular to the two lines. Then the equations of the other line may be written $z = k$, $Y = mx$. If (x, y) are the coördinates in the rotating plane of its intersection with the other line, the origin being the same as for the space coördinates, then $y^2 - Y^2 = z^2$ or $y^2 - m^2x^2 = k^2$. Hence the curve is an hyperbola unless the two lines are perpendicular, and in this case the curve reduces to two coincident straight lines. Both branches of the hyperbola are traced in a complete revolution of the plane.

Note by the Editors: The above results are evident without the use of equations, for we may suppose that the plane is fixed and that the other line rotates about the first line, fixed in the plane. The rotating line describes an hyperboloid of one sheet, and, since the plane passes through the axis of the hyperboloid, it cuts out an hyperbola.

Also solved by A. Bogard, H. C. Bradley, William Hoover, W. J. Patterson, A. Pelletier, F. L. Wilmer and the Proposer.

2978 [1922, 271]. Proposed by C. N. SCHMALL, New York City.

Given the equation of the general cubic, $f(x, y) \equiv ax^3 + 3bx^2y + 3cxy^2 + dy^3 + a_1x^2 + 2b_1xy + c_1y^2 + a_2x + b_2y + c_3 = 0$, show that the three asymptotes of the curve will be concurrent if

$$\begin{vmatrix} a & b & a_1 \\ b & c & b_1 \\ c & d & c_1 \end{vmatrix} = 0.$$

Solution by J. B. Reynolds, Lehigh University.

Let $y = sx + k$ be an asymptote. Substituting this in the equation of the curve we find the equations of condition for infinite roots to be

$$a + 3bs + 3cs^2 + ds^3 = 0, \tag{1}$$

$$3k(b + 2cs + ds^2) + (a_1 + 2b_1s + c_1s^2) = 0. \tag{2}$$

The equation of the asymptote may then be written

$$3(b + 2cs + ds^2)y - 3(bs + 2cs^2 + ds^3)x + (a_1 + 2b_1s + c_1s^2) = 0,$$

and this equation by aid of (1) as

$$3(a + 2bs + cs^2)x + 3(b + 2cs + ds^2)y + (a_1 + 2b_1s + c_1s^2) = 0. \tag{3}$$

If s_1, s_2, s_3 are the roots of (1), then the determinant of the three equations obtained by inserting these roots in (3) is readily seen to be the product

$$18 \begin{vmatrix} 1 & s_1 & s_1^2 \\ 1 & s_2 & s_2^2 \\ 1 & s_3 & s_3^2 \end{vmatrix} \begin{vmatrix} a & b & a_1 \\ b & c & b_1 \\ c & d & c_1 \end{vmatrix}$$

and hence the theorem of the problem follows at once.

Another way of obtaining the result is as follows: If the determinant in the problem is zero, then the three lines

$$3ax + 3by + a_1 = 0,$$
$$3bx + 3cy + b_1 = 0,$$
$$3cx + 3dy + c_1 = 0$$

meet in a point. Now (3) is a linear combination of these three lines and hence passes through the same point for all values of s.

Also solved by A. Bogard, H. A. DoBell, R. E. Gaines, A. M. Harding, William Hoover, A. V. Richardson, Hazel E. Shoemaker, F. L. Wilmer and the Proposer.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Professor WILLIAM CAIN, of the University of North Carolina, has been awarded the James R. Croes medal of the American Society of Civil Engineers for his paper on *The circular arch under normal loads.*

Assistant Professor G. W. MULLINS, of Barnard College, Columbia University, has been promoted to an associate professorship of mathematics.

Assistant Professor F. N. BRYANT, of Syracuse University, has been promoted to a full professorship of mathematics.

Mr. B. F. KIMBALL has been appointed instructor of mathematics at Tulane University.

Announcement is made that the Council of the American Mathematical Society has decided to proceed to raise an endowment fund of at least one hundred thousand dollars and a Committee on Endowment has been appointed, consisting of Julian L. Coolidge, Harvard University (Chairman); Arnold Dresden, University of Wisconsin; Griffith C. Evans, Rice Institute; Robert Henderson, Equitable Life Assurance Society; and George E. Roosevelt, 30 Pine Street, New York (Treasurer). While recognizing that there are many other worthy objects, it has been voted that income may be used for publishing of material pertaining to research, such as the transactions and other journals, the colloquium lectures, and books and treatises. The release of a portion of the regular income will permit the Society to attend to some of the other pressing matters.

A campaign for a wider circulation and for an extension of its influence is now being conducted by "SCIENTIA," the international review published at Milan, Italy. This journal aims to promote the work of the philosophic synthesis of science and the intellectual fraternization of peoples. Its authoritative but non-technical articles (each published simultaneously in the language of the author and in French) give a summary and evaluation of scientific truth not found elsewhere. As such a journal necessarily makes its appeal to a select group of intellectual people, it can succeed in its aims only if supported by all those who believe in the formation of an international scientific mind infused with the spirit of a philosophic interpretation of scientific truth. American subscribers can purchase "SCIENTIA" through Messrs. Williams & Wilkins, Mount Royal and Guilford Avenues, Baltimore, Md., at ten dollars per year (twelve issues). Among contributors to recent or forthcoming issues are the following Americans: W. D. MacMillan, S. Nearing, L. E. Dickson, W. S. Adams, and R. D. Carmichael.

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